UNIVERSITY
OF SKÖVDE

School of Engineering

WRITTEN EXAMINATION

Course: Mechanics IV

Examination: Sub-course

Course code: MT355G | Credits for written examination: 3 hp

Date: 2025-11-21 | Examination time: 8:15-12:30

Examination responsible: Karl Mauritsson
Teachers concerned: Karl Mauritsson, Mahdi Eynian, Daniel Svensson
Aid at the exam/appendices:

- The course book: Inman D. J. (2014). Engineering Vibrations. (4th ed) Essex England: Pearson. ISBN 9780273768449.
- Formulae Sheet
- Råde, L, Westergren, B. (1990). Beta – Mathematics Handbook. Lund: Studentlitteratur.
- Sundström, B. (red.) (2010). Handbook of Solid Mechanics. Stockholm: Department of Solid Mechanics, KTH. ISBN 9789197286046. Or the Swedish version

- Sundström, B. (1999). Handbok och formelsamling i hållfasthetslära. Tekniska högskolan Stockholm: Institution för hållfasthetslära.
- An approved calculator according to "Allmänna riktlinjer gällande utbildning på Institutionen for ingenjörsvetenskap": Detailed list of calculators is provided to the exam invigilators.
- Casio Teknikräknare FX-82 all variants
- Texas Instruments Tl-30 all variants
- Texas Instruments Tl-82, Tl-83, Tl-84
- Casio FX-7400Gii, Fx-9750GII

The exam invigilators can provide a scientific calculator during the exam if you need one.

An English-Swedish-English or English-Spanish-English dictionary.
No added notes are allowed in the texts used during the examination.

The solution of each problem of the written exam is first assessed and graded based on the following guidelines:

PointsGeneral description
5 EXCELLENT - outstanding results with only minor deficiencies. High level of knowledge. Good analytical ability. Can use the knowledge independently.
4 VERY GOOD - above average standards but with some shortcomings. Good overview of the field of knowledge. Can use the knowledge independently
3 GOOD - generally good work with some shortcomings. Can account for the most important parts of the subject. Can largely use the knowledge independently.
2 SATISFACTORY - pretty good but with significant shortcomings. Can account for the most important parts of the subject. Can to some extent use the knowledge independently
1 SUFFICIENT - The result meets the minimum requirements but not more. The overview of the most important parts of the subject is inadequate. To a limited extent, use the knowledge independently.
0 FAIL - more work required before credit can be given
If any of the three problems is graded fail (0), the written exam is graded fail (F). Otherwise, the final grade is given by the sum points according to:

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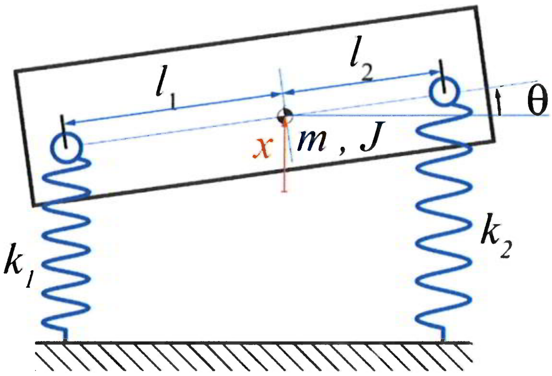
Question 1 (5 p)

The figure shows a simplified dynamic model of a car and its suspension springs. Linear and angular acceleration equations are obtained as:

$$\begin{cases} m\ddot{x} &= -k_1(x - l_1\theta) - k_2(x + l_2\theta) \\ J\ddot{\theta} &= k_1 l_1(x - l_1\theta) - k_2 l_2(x + l_2\theta) \end{cases}$$

values of mass, moment of inertia, lengths, and stiffness are given as:

$m = 625\text{kg}$, $J = 400\text{kgm}^2$, $l_1 = 2.0$ m, $l_2 = 1.0$ m, $k_1 = k_2 = 30$ kN/m



1. Obtain the modal natural frequencies of the system.
2. Propose values (in N.s/m) for dampers c_1 and c_2 parallel to k_1 and k_2 respectively such that damping ratio of the second mode becomes 0.2 [=20%]. Assume proportional damping, with $\alpha = 0$. Evaluate the damping ratio of the first mode with these values.
3. Calculate how much **amplitude** of each mode becomes after $t = 5T_1$(compared to its initial value), after both modes get excited due to an impulse at $t = 0$.
   T_1 is the period of the first vibration mode.

[**Hint:** You may not need to extract the mode shapes/eigenvectors.]

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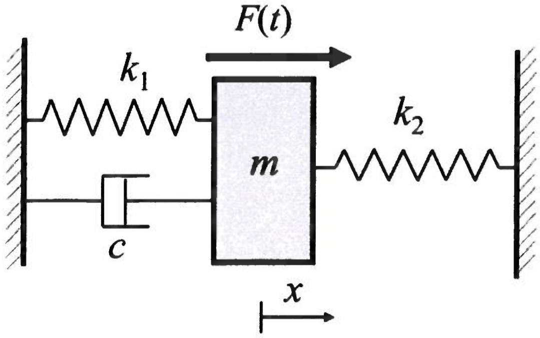
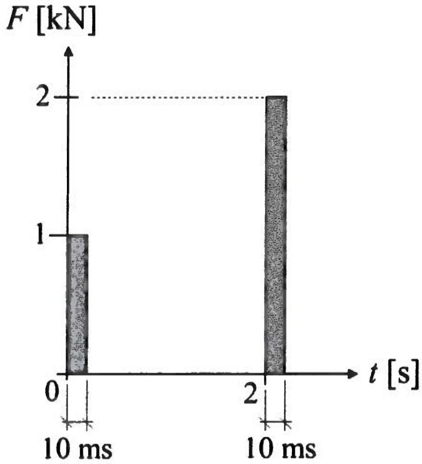
Question 3 (5 p)

A spring-mass-damper system is initially at rest, but is disturbed by two impulse loads. At time $t = 0$, a force of magnitude 1 kN hits the mass and acts for 10 ms. At time $t = 2$ s, a force of magnitude 2 kN hits the mass and acts for 10 ms. The system parameters are:
$m = 10$ kg, $k_1 = 400$ N/m , $k_2 = 600$ N/m , $c = 100$ kg/s.





Calculate the response $x(t)$ for $t \geq 0$.

1.2.3 Overdamped case ($\zeta > 1$)

$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-(\omega_n\sqrt{\zeta^2-1})t} + a_2 e^{+(\omega_n\sqrt{\zeta^2-1})t}\right)$	$a_1 = \dfrac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$	$a_2 = \dfrac{v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$
Eq. 1.41	Eq. 1.42	Eq. 1.43

2 Forced Vibration ($F(t) \neq 0$)

The total solution $x(t)$ is always the sum of the **particular solution,** $x_p(t)$ with the frequency of driving force, added to the **homogenous solution,** $x_h(t)$ with natural frequency of the system, with similar to equations to the free vibration (see section 1) but NOT the same constants as the free vibration, in other words: $x(t) = x_h(t) + x_p(t)$.

- The coefficients of the **homogenous solution** are adjusted such that the total solution satisfies the initial conditions.
- In damped systems, after a while, the response from the initial conditions will die out and the system's vibration will be dominated by the particular response (solution).

2.1 Harmonic excitation $F(t) = F_0\cos(\omega t)$ or $f(t) = \frac{F(t)}{m} = f_0\cos(\omega t)$

($f_0 = \frac{F_0}{m}$, note that the SI unit for f_0 is $\left[\frac{\mathrm{N}}{\mathrm{kg}}\right] = \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$).

2.1.1 Undamped Case (c = 0, $\zeta = 0, r \neq 1$)

Differential equation: $\ddot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution: $x_p(t) = \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$

Total solution with IC: $x(t) = \overbrace{\frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos\omega_n t}^{x_h(t)} + \overbrace{\frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)}^{x_p(t)}$ (eq. 2.11)

Zero initial conditions in this case will lead to beating, with amplitude $\left|\frac{2f_0}{\omega_n^2-\omega^2}\right|$ and beat frequency of $\omega_{beat} = |\omega_n - \omega|$

$$x(t) = \frac{2f_0}{\omega_n^2-\omega^2}\sin\left(\frac{\omega_n-\omega}{2}t\right)\sin\left(\frac{\omega_n+\omega}{2}t\right) \text{(eq. 2.13)}$$

2.1.2 Resonance at undamped case ($c = 0, \omega = \omega_n$) or ($\zeta = 0, r = 1$)

$x(t) = A_1\sin\omega t + A_2\cos\omega t + \frac{f_0}{2\omega}t\sin\omega t$ (eq. 2.17), A_1, A_2 depend on initial conditions.

2.1.3 Damped Case

Differential equation: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution (for **both** $\zeta < 1$ and $\zeta \geq 1$): $x_p(t) = X\cos(\omega.t - \theta)$

$$X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega.\omega_n)^2}} = \frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}}.\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

and $\frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}} = \frac{F_0}{m\omega_n^2} = \frac{F_0}{k}$ (i.e. displacement of the spring if F_0 was applied statically)

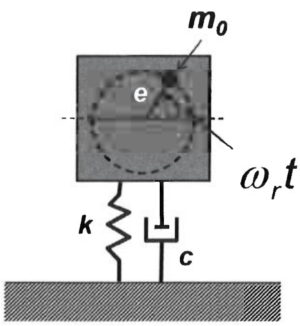
$$\theta = \text{atan2}(\ 2\zeta\omega_n\omega,\ \ \omega_n^2 - \omega^2) = \textbf{atan2}(2\,\zeta\, r,\ \ 1 - r^2)$$

2.1.3.1 Resonance (for $\mathbf{0} \leq \zeta \leq \frac{1}{\sqrt{2}}$)

$$\frac{\omega_{peak}}{\omega_n} = r_{peak} = \sqrt{1-2\zeta^2}$$

$$X_{peak} = \frac{f_0}{\omega_n^2}.\frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{2k\zeta\sqrt{1-\zeta^2}} \overbrace{\cong}^{\text{if } \zeta \ll 1} \frac{F_0}{2k\zeta}$$

(m is the total mass of the machine, including the unbalance mass. m_0 is the unbalanced mass, that rotates with eccentricity e and angular velocity of ω_r).



Particular solution: $$x_p(t) = X\sin(\omega_r t - \theta)$$

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega_r^2)^2 + (2\zeta\omega_r\omega_n)^2}} = e.\frac{m_0}{m}.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega_r}{\omega_n}$$

$$\theta = \text{atan2}(2\zeta\omega_n\omega_r \quad , \quad \omega_n^2 - \omega_r^2) = \text{atan2}(2\,\zeta\, r \quad , \quad 1 - r^2)$$

$\frac{Xm}{em_0} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ is a very small number at low frequencies, at resonance it becomes almost $\frac{1}{2\zeta}$ and at very high frequencies it becomes 1 (with $\theta \cong \pi$), this means the machine $(m - m_0)$ moves in the opposite direction, to keep the center of total mass in an almost stationary position.

5 Linear Systems, Superposition

1. For a linear homogeneous differential equation, e.g. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$:
- if $x_1(t)$ and $x_2(t)$ are [homogenous] solutions to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$,

  then $a_1x_1(t) + a_2x_2(t)$ is a [homogenous] solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$.

2. For a linear equation of motion, e.g. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ with constant coefficients for $\ddot{x}, \dot{x}, x$:
- if $x_1(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_1(t)$,
- and if $x_2(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_2(t)$,

  then $a_1x_1(t) + a_2x_2(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a_1f_2(t) + a_2f_2(t)$.

6 Response to a Periodic Excitation (Fourier Series)

Any periodic function $F(t)$ with period T could be represented by an infinite series of the form:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\ cos(n\omega_T t) + b_n\, sin(n\omega_T t)]$$

With $\omega_T = \frac{2\pi}{T}$, $a_0 = \frac{2}{T}\int_0^T F(t)dt$, $a_n = \frac{2}{T}\int_0^T F(t)\,cos(n\omega_T t)\,dt$ and $b_n = \frac{2}{T}\int_0^T F(t)\,sin(n\omega_T t)\,dt$. (Eq. 3-20 to 3.23). The superposition principle could be used to calculate the response to the periodic force by calculating the response to each Fourier term and adding the resulting displacements.

7 Response to impulse excitation, underdamped SDOF:

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t - \tau)$$
$$\Rightarrow x(t) = \hat{F}.h(t - \tau)$$

$$h(t-\tau) = \frac{1}{m\omega_d}.e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau) \qquad t \geq \tau \qquad \text{(eq. 3. 9)}$$

8 Response to arbitrary excitation

$$x(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t[F(\tau).e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)]d\tau = \frac{1}{m\omega_d}\int_0^t[F(t-\tau).e^{-\zeta\omega_n\tau}\,sin\,\omega_d\tau]d\tau \qquad (3.13)$$

then the result of transformation $\tilde{\mathbf{C}} = \mathbf{L}^{-1}\mathbf{C}(\mathbf{L}^{-1})^{\mathrm{T}} = \alpha \mathrm{I} + \beta\tilde{\mathbf{K}}$ becomes diagonal if the matrix of eigenvectors of $\tilde{\mathbf{K}}$ are multiplied to it from the right ($\mathbf{P}$) and left ($\mathbf{P}^{\mathrm{T}}$) as follows:

$$\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$$

Replacing $\mathbf{x}(t)$ with $\mathbf{x}(t) = (\mathbf{L}^{-1})^{\mathbf{T}}\mathbf{q}(t)$ in the differential equation (4.126) and multiplying $\mathbf{L}^{-1}$ from left results in:

$\mathbf{I}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{q}}(t) + \tilde{\mathbf{K}}\mathbf{q}(t) = \mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ *(similar to eq. 4.128)*

Defining $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$, where $\mathbf{P}$ is the orthonormal eigenvector matrix of $\tilde{\mathbf{K}}$, [note that this results in $\mathbf{x}(t) = (\mathbf{L}^{-1})^{T}\ \mathbf{q}(t) = (\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{P}\,\mathbf{r}(t)$ and With $\mathbf{S} = (\mathbf{L}^{-1})^{T}\ \mathbf{P}$ and $\mathbf{S}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$ then $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(t)$ and $\mathbf{r}(t) = \mathbf{S}^{-1}\mathbf{x}(t)$]

replacing $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$ in *(eq. 4.128)* multiplying $\mathbf{P}^{\mathrm{T}}$from left to this equation results in:

$\mathbf{I}_{n\times n}\ddot{\mathbf{r}}(t) + \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]\,\dot{\mathbf{r}}(t) + \mathbf{\Lambda}\mathbf{r}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ *(similar to eq. 4.129)*

In *above equation:*

- $\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$ and $\mathbf{\Lambda} = \mathbf{P}^{\mathrm{T}}\tilde{\mathbf{K}}\,\mathbf{P} = \mathrm{diag}(\boldsymbol{\omega}_{\mathrm{i}}^{2})$
- The vector $\mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t) = \mathbf{S}^{\mathrm{T}}\mathbf{B}\mathbf{F}(t)$ has elements $f_i(t)$ that will be linear combination of forces applied to the degrees of freedom.
- The modal initial conditions are calculated as $\mathbf{r}(\mathbf{0}) = \mathbf{S}^{-1}\mathbf{x}_{\mathbf{0}}$ and $\dot{\mathbf{r}}(\mathbf{0}) = \mathbf{S}^{-1}\dot{\mathbf{x}}_{\mathbf{0}}$
- The response for each mode (elements of $\mathbf{r}(t)$) could be calculated similar to the response of single degree of freedom systems with $f_i(t)$ excitation, these equations are called **modal equations**:

$$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_{\mathrm{i}}^{2}r_i(t) = f_i(t)$$

(e.g. if it is harmonic excitation by the same equations as in 2.1), . or by eq. 3.13.

The resulting $r_i(t)$s are assembled back in $\mathbf{r}(t)$.

- The response in natural coordinate system is obtained by $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(\mathrm{t})$
- Since $\mathbf{S}^{\mathbf{T}}\mathbf{M}\mathbf{S} = \mathbf{I}_{n\times n}$, columns of $\mathbf{S} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots]$ are mass-normalized mode shapes of the system.

Physical, Mass Normalized and Modal Spaces with [SI units] (for translational mass systems):

Eq.	Name	Mass Matrix	Damping Matrix	Stiffness Matrix	Matrix Transformation	State Vector	State Vector T	
							↓	
$\mathbf{M}[\mathrm{kg}]\ddot{\mathbf{x}}\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] + \mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right]\dot{\mathbf{x}}\left[\frac{\mathrm{m}}{s}\right] + \mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]\mathbf{x}[\mathrm{m}] = \mathbf{BF}(t)\ [\mathrm{N}]$	Physical Space	$\mathbf{M}[\mathrm{kg}]$	$\mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right]$	$\mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$		$\mathbf{X}(t)[m]$		$\mathbf{X}(t$ $= \mathbf{S}$
$\mathbf{I}\ddot{\mathbf{q}}\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right] + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \mathbf{L}^{-1}\mathbf{BF}(t)\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]$	Mass Normalized	$\mathbf{I}[-]$	$\tilde{\mathbf{C}}\left[\frac{\mathrm{N.s}}{\mathrm{kg.m}} = \frac{1}{\mathrm{s}}\right]$	$\tilde{\mathbf{K}}\left[\frac{\mathrm{N}}{\mathrm{kg.m}} = \frac{1}{\mathrm{s}^2}\right]$	$\tilde{\mathbf{K}} = (\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T$	$\mathbf{q}(t)[\mathrm{m}\sqrt{\mathrm{kg}}]$	$\mathbf{q}(t) = \mathbf{L}^{\mathrm{T}}\mathbf{X}(t)$	$\mathbf{q}($
$\ddot{\mathbf{r}}\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right] + \mathrm{diag}[2\zeta_i\omega_i]\dot{\mathbf{r}} + \Lambda\mathbf{r} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{BF}(t)\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]$ Decoupled differential (known as **Modal Equations**). equation*, for $i = 1\ to\ n$: $\ddot{r}_i + 2\zeta_i\omega_i\dot{r}_i + \omega_i^2 r_i = f_i(t)$	Modal Space	$\mathbf{I}[-]$	$[\mathrm{diag}(2\zeta_i\ \omega_i)] = \left[\frac{1}{\mathrm{s}}\right]$ (*)	$\Lambda\left[\frac{1}{\mathrm{s}^2}\right] = [\mathrm{diag}(\omega_i^2)]$	$\Lambda = \mathbf{P}^{\mathrm{T}}\tilde{\mathbf{K}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T\mathbf{P} = \mathbf{S}^{\mathrm{T}}\mathbf{K}\,\mathbf{S}$	$\mathbf{r}(t)[\mathrm{m}\sqrt{\mathrm{kg}}]$	$\mathbf{r}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{q}(t) = \mathbf{S}^{-1}\mathbf{X}(t)$	
* ONLY IF $\mathbf{S}^{\mathrm{T}}\mathbf{C}\,\mathbf{S}$ becomes a diagonal matrix $[\mathrm{diag}(2\zeta_i\ \omega_i)]$, e.g. when $\mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right] = \alpha\left[\frac{1}{\mathrm{s}}\right]\mathbf{M}[\mathrm{kg}] + \beta[\mathrm{s}]\,\mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$, then $\mathbf{S}^{\mathrm{T}}\mathbf{C}\,\mathbf{S} = \alpha\mathbf{I} + \beta\Lambda = [diag(2\zeta_i\ \omega_i\ )$								

Transformation Matrices:

	Description	Definition	Calculation in MATLAB
$\mathbf{L}[\sqrt{\mathrm{kg}}]$	Normalization of Mass Matrix Lower triangular Cholesky's matrix for **M**	$\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$	`L = chol(M, 'lower');`
$\mathbf{P}[-]$	Makes $\tilde{\mathbf{K}}$ diagonal	$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots\ ]$ Matrix of Orthonormal Eigenvectors of $\tilde{\mathrm{K}}$ $\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I}$	`K_tilde = (L^(-1))*K*(L^(-` `[P, Lambda] = eig(K_tilde`

$\mathbf{S}\left[\frac{1}{\sqrt{\mathrm{kg}}}\right]$	Matrix of Mode Shapes,
	Moves from Modal Space to Physical Space
		$\mathbf{S} = (\mathbf{L}^{-1})^{\mathbf{T}}\mathbf{P}$
		Also $\mathbf{S}^{-1}[\sqrt{\mathrm{kg}}] = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$ and
		$\mathbf{S}^{\mathrm{T}}\left[\frac{1}{\sqrt{\mathrm{kg}}}\right] = \mathbf{P}^{\mathrm{T}}(\mathbf{L}^{-1})$
		(in general, $\mathbf{S}^{\mathbf{T}} \neq \mathbf{S}^{-1}$)
			`S = (L^(-1))' * P`
			`% Or`
			`S = (L')\P`

14 Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

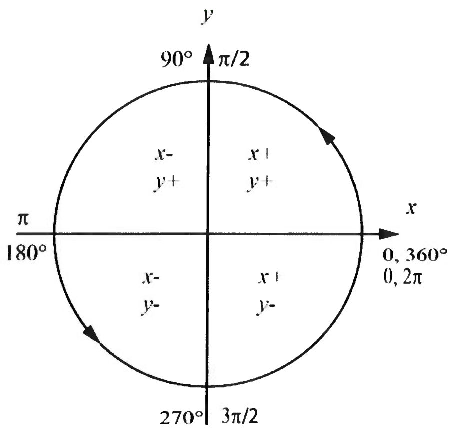
15 Quadratic equation

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Reduced form (a = 1):

$$x^2 + px + q = 0 \Rightarrow x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

16 Four Quadrant Arctangent Function, $\operatorname{atan2}(y, x)$



$$\operatorname{atan2}(y,x) = \begin{cases} \operatorname{atan}\left(\frac{y}{x}\right) & x > 0 \\ \operatorname{atan}\left(\frac{y}{x}\right) + \pi & y \geq 0; x < 0 \\ \operatorname{atan}\left(\frac{y}{x}\right) - \pi & y < 0; x < 0 \\ \frac{\pi}{2} & y > 0; x = 0 \\ -\frac{\pi}{2} & y < 0; x = 0 \\ \text{undefined} & y = x = 0 \end{cases}$$

In MATLAB and many other software, the correct form is atan2(y,x), **but** in Excel, you should enter ATAN2(x;y) to get the correct answer.