UNIVERSITY
OF SKÖVDE

School of Engineering Science

WRITTEN EXAMINATION

Course: Mekanik IV / Mechanics IV

Sub-course

Course code: MT355G Credits for written examination: 3 hp

Date: 2025-01-11 Examination time: 09:15-13:30

Examination responsible: Karl Mauritsson

Teachers concerned: Karl Mauritsson, Mahdi Eynian, Daniel Svensson

Aid at the exam/appendices

• Inman D. J. (2014). Engineering Vibrations. (4th ed) Essex England: Pearson. ISBN 9780273768449 or printed pages from the e-book version

• Råde, L, Westergren, B. (1990). *Beta – Mathematics Handbook*. Lund: Studentlitteratur.
Or a similar handbook

• Sundström, B. (red.) (2010). *Handbook of Solid Mechanics.* Stockholm: Department of Solid Mechanics, KTH. ISBN 9789197286046.
Or the Swedish version

• Sundström, B. (1999). *Handbok och formelsamling i hållfasthetslära.* Tekniska högskolan Stockholm: Institution för hållfasthetslära.

• An approved calculator according to "Allmänna riktlinjer gällande utbildning på Institutionen for ingenjörsvetenskap":
- o Casio Teknikräknare FX-82 all variants
- o Texas Instruments Tl-30 all variants
- o Texas Instruments Tl-82, Tl-83, Tl-84
- o Casio FX-7400Gii, Fx-9750GII

• An English-Swedish-English ordbok or English-Spanish-English dictionary.

No added notes are allowed in the texts used during the examination.

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Instructions	☐	Take a new sheet of paper for each teacher.
	☒	Take a new sheet of paper when starting a new question.
	☒	Write only on one side of the paper.
	☒	Write your name and personal ID No. on all pages you hand in.
	☒	Use page numbering.
	☒	Don´t use a red pen.
	☒	Mark answered questions with a cross on the cover sheet.

Examination results should be made public within 18 working days

Good luck!

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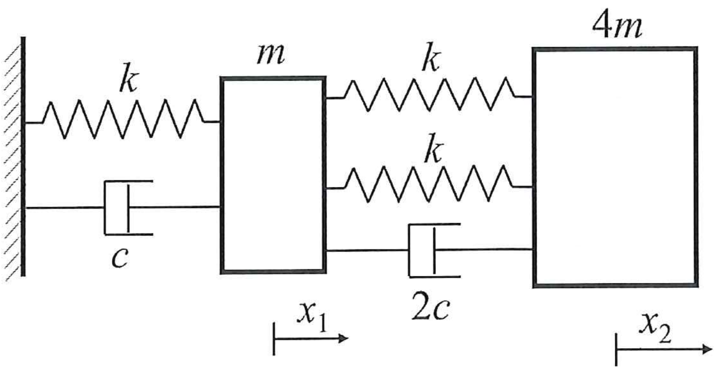
Question 1 (5 p)

The mass-spring system below is started at its static equilibrium position $x_1(0) = x_2(0) = 0$ with initial velocities $\dot{x}_1(0) = \dot{x}_2(0) = 1$ m/s. The following dynamic parameters are given:

$k = 2$ kN/m, $c = 50$ kg/s, $m = 100$ kg

Calculate the response (x_1 and x_2 as functions of time).



Question 2 (5 p)

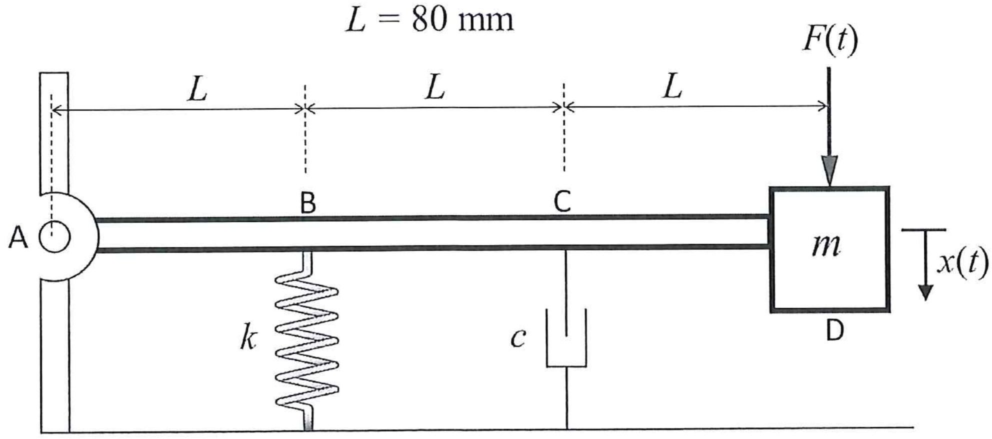
A foot pedal for a musical instrument is modeled according to the figure. The applied load $F(t)$ is a harmonic force with an amplitude of 60 N and a driving frequency of 1 Hz.

The mass of the lever is negligible. The dynamic parameters of the model are:

$k = 2500$ N/m, $c = 50$ kg/s, $m = 40$ kg

Compute the steady state response, described as the vertical deflection of the mass $x(t)$.

You can assume that the angle that the pedal rotates around A is small. Hence, points B, C and D are only moving in the vertical direction.



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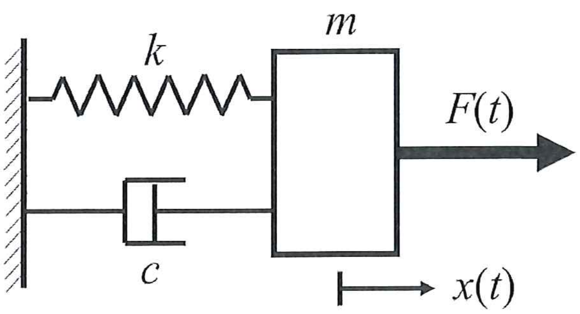
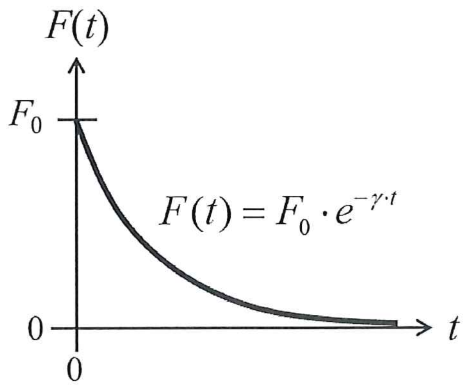
Question 3 (5 p)

A mass-spring system starts at rest and $x(0) = 0$ when a force $F(t)$ is applied. The force decays exponentially over time, with the initial magnitude F_0 and the decay rate constant γ.

The dynamic parameters of the system are m, k and $c = \frac{2}{3}\sqrt{km}$.

Calculate the response $x(t)$. How large is the response after a long time?





Some integral from the table below may be useful.

Integral table

$$\int sin(ax)dx = -\frac{1}{a} \cdot cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a} \cdot sin(ax)$$

$$\int x \cdot sin(ax)dx = \frac{1}{a^2}(sin(ax) - ax \cdot cos(ax))$$

$$\int x \cdot cos(ax)dx = \frac{1}{a^2}(cos(ax) + ax \cdot sin(ax))$$

$$\int e^{ax} \cdot sin(bx)dx = \frac{e^{ax}}{a^2 + b^2}(a \cdot sin(bx) - b \cdot cos(bx))$$

$$\int e^{ax} \cdot cos(bx)dx = \frac{e^{ax}}{a^2 + b^2}(a \cdot cos(bx) + b \cdot sin(bx))$$

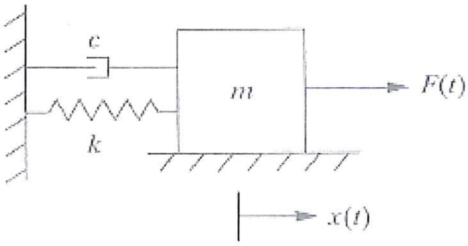
General Case:

$m\ddot{x} + c\dot{x} + kx = F(t)$ Newton's 2nd Law (NSL)

with $\zeta = \frac{c}{2\sqrt{k.m}}, \omega_n = \sqrt{k/m}, f(t) = F(t)/m$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ Mass normalized form

initial conditions: $x(0) = x_0$ and $\dot{x}(0) = v_0$

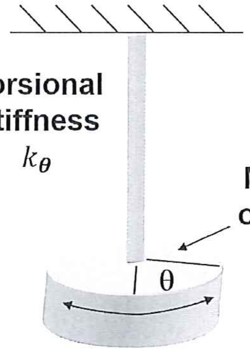


Dynamics of rotating systems

$$J[\mathrm{kg\cdot m^2}]\,\ddot{\theta} + c_\theta\left[\frac{\mathrm{N\cdot m}}{\mathrm{s}}\right]\dot{\theta} + k_\theta[\mathrm{N\cdot m}]\theta = T(t)[\mathrm{N\cdot m}]$$

with $\zeta = \frac{c_\theta}{2\sqrt{k_\theta.J}}, \omega_n = \sqrt{\frac{k_\theta}{J}}, f(t) = T(t)/J$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)$$



1 Free Vibration ($F(t) = 0$)

1.1 Undamped Case ($c = 0$)

$$\ddot{x} + \omega_n^2 x = 0$$

solution:

$x(t) = a.\sin(\omega_n t) + b.\cos(\omega_n t)$	$a = \frac{v_0}{\omega_n}$	$b = x_0$
	Or	
$x(t) = A.\sin(\omega_n t + \phi)$	$A = \frac{\sqrt{x_0^2\omega_n^2 + v_0^2}}{\omega_n}$	$\phi = \mathbf{atan2}(x_0\omega_n, \quad v_0)$ [1]

1.2 Damped Case (With Viscous Damping, $c \neq 0$)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \zeta = \frac{c}{2\sqrt{k.m}} \neq 0$$

1.2.1 Underdamped case ($0 < \zeta < 1$)

solution:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$x(t) = e^{-\zeta\omega_n t}[a.\sin(\omega_d t) + b.\cos(\omega_d t)]$	$a = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$	$b = x_0$
	Or	
$x(t) = Ae^{-\zeta\omega_n t}.\sin(\omega_d t + \phi)$	$A = \frac{\sqrt{\omega_d^2 x_0^2 + (v_0^2 + \zeta\omega_n x_0)^2}}{\omega_d}$	$\phi = \mathbf{atan2}(\omega_d x_0, \quad v_0 + \zeta\omega_n x_0)$

1.2.2 Critically damped case ($\zeta = 1$)

$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$	$a_1 = x_0$	$a_2 = v_0 + \omega_n x_0$
Eq. 1.45	Eq. 1.46	Eq. 1.46

[1] $\mathbf{atan2}(y, x)$ indicates the four quadrant inverse tangent ($\mathbf{arctan}$), for example atan2(-1,-1) = -3π/4 , while **atan**(-1/-1) = π/4, see page 10.

1.2.3 Overdamped case ($\zeta > 1$)

$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-(\omega_n\sqrt{\zeta^2-1})t} + a_2 e^{+(\omega_n\sqrt{\zeta^2-1})t}\right)$	$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$	$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$
Eq. 1.41	Eq. 1.42	Eq. 1.43

2 Forced Vibration ($F(t) \neq 0$)

The total solution $x(t)$ is always the sum of the **particular solution,** $x_p(t)$ with the frequency of driving force, added to the **homogenous solution,** $x_h(t)$ with natural frequency of the system, with similar to equations to the free vibration (see section 1) but NOT the same constants as the free vibration, in other words: $x(t) = x_h(t) + x_p(t)$.

- The coefficients of the **homogenous solution** are adjusted such that the total solution satisfies the initial conditions.
- In damped systems, after a while, the response from the initial conditions will die out and the system's vibration will be dominated by the particular response (solution).

2.1 Harmonic excitation $F(t) = F_0\cos(\omega t)$ or $f(t) = \frac{F(t)}{m} = f_0\cos(\omega t)$

($f_0 = \frac{F_0}{m}$, note that the SI unit for f_0 is $\left[\frac{\text{N}}{\text{kg}}\right] = \left[\frac{\text{m}}{\text{s}^2}\right]$).

2.1.1 Undamped Case (c = 0, $\zeta = 0, r \neq 1$)

Differential equation: $\ddot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution: $x_p(t) = \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$

Total solution with IC: $x(t) = \overbrace{\frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos\omega_n t}^{x_h(t)} + \overbrace{\frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)}^{x_p(t)}$ (eq. 2.11)

Zero initial conditions in this case will lead to beating, with amplitude $\left|\frac{2f_0}{\omega_n^2-\omega^2}\right|$ and beat frequency of $\omega_{beat} = |\omega_n - \omega|$

$$x(t) = \frac{2f_0}{\omega_n^2-\omega^2}\sin\left(\frac{\omega_n-\omega}{2}t\right)\sin\left(\frac{\omega_n+\omega}{2}t\right)\text{(eq. 2.13)}$$

2.1.2 Resonance at undamped case ($c = 0, \omega = \omega_n$) or ($\zeta = 0, r = 1$)

$x(t) = A_1\sin\omega t + A_2\cos\omega t + \frac{f_0}{2\omega}t\sin\omega t$ (eq. 2.17), A_1, A_2 depend on initial conditions.

2.1.3 Damped Case

Differential equation: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution: $x_p(t) = X\cos(\omega.t - \theta)$

$$X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega.\omega_n)^2}} = \frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}}.\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

and $\frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}} = \frac{F_0}{m\omega_n^2} = \frac{F_0}{k}$ (i.e. displacement of the spring if F_0 was applied statically)

$$\theta = \text{atan2}(\ 2\zeta\omega_n\omega,\ \ \omega_n^2 - \omega^2) = \textbf{atan2}(2\,\zeta\, r,\ \ 1 - r^2)$$

2.1.3.1 Resonance (for $0 \le \zeta \le \frac{1}{\sqrt{2}}$)

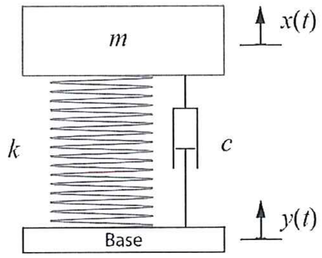
$$\frac{\omega_{peak}}{\omega_n} = r_{peak} = \sqrt{1-2\zeta^2}$$

$$X_{peak} = \frac{f_0}{\omega_n^2}.\frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{2k\zeta\sqrt{1-\zeta^2}} \overbrace{\cong}^{\text{if } \zeta \ll 1} \frac{F_0}{2k\zeta}$$

3 Base Excitation

3.1 Harmonic Excitation

$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0, \quad y = Y\sin(\omega t)$ from NSL



Standard form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega Y \times \cos(\omega t) + \omega_n^2 Y \times \sin(\omega t)$$

Particular solution:

$$x_p(t) = X\,.\sin(\omega.t - \psi)$$

$$X = Y.\sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

Or

$$X = Y.\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

$$\psi = \mathrm{atan2}[mc\omega^3 \quad , \quad k(k - m\omega^2) + (c.\omega)^2] = \mathrm{atan2}[2\zeta r^3 \quad , \quad (1 - r^2) + (2\zeta r)^2]$$

$\left|\frac{X}{Y}\right| = \sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **displacement transmissibility,** it is around 1 at low frequencies, then it reaches to its maximum (resonance) very close to $r = 1$, to a value close to $\frac{1}{2\zeta}$. This ratio reduces and reaches 0 as the r increases (i.e. when base vibration frequency increases to values much higher than the natural frequency, the mass remains almost still. In other words, you cannot oscillate an object at frequencies much higher than the natural frequency that is created between that object and its base). (Fig. 2.14)

3.1.1 Transmitted Force

$$F(t) = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x} = -m\omega^2 X.\sin(\omega.t - \psi)$$

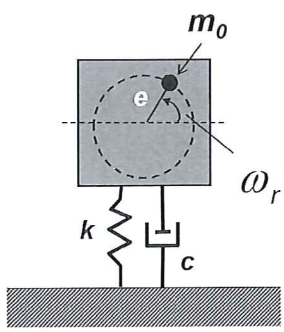
$$\left|\frac{F}{Y}\right| = m\omega^2\left|\frac{X}{Y}\right| = (k.r^2) \times \sqrt{\frac{1 + (2.\zeta.r)^2}{(1 - r^2)^2 + (2.\zeta.r)^2}}.$$

$\left|\frac{F}{kY}\right| = r^2 \times \sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **force transmissibility ratio (FTR)**. This ratio is very small at low frequencies (compared to the natural frequency). It has a local peak very close to $r = 1$, approaching $\frac{1}{2\zeta}$. With non-zero damping ratio, **FTR** keeps increasing as the r ratio increases. (Fig. 2.15 in the book).

4 Rotating Unbalance

NSL: $\quad m\ddot{x} + c\dot{x} + kx = m_0 e\omega_r^2 \sin(\omega_r\, t) = F_0 \sin(\omega_r\, t)$

(m is the total mass of the machine, including the unbalance mass. m_0 is the unbalanced mass, that rotates with eccentricity e and angular velocity of ω_r).



Particular solution: $$x_p(t) = X\sin(\omega_r t - \theta)$$

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega_r^2)^2 + (2\zeta\omega_r\omega_n)^2}} = e.\frac{m_0}{m}.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega_r}{\omega_n}$$

$$\theta = \text{atan2}(2\zeta\omega_n\omega_r \quad , \quad \omega_n^2 - \omega_r^2) = \text{atan2}(2\,\zeta\, r \quad , \quad 1 - r^2)$$

$\frac{Xm}{em_0} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ is a very small number at low frequencies, at resonance it becomes almost $\frac{1}{2\zeta}$ and at very high frequencies it becomes 1 (with $\theta \cong \pi$), this means the machine $(m - m_0)$ moves in the opposite direction, to keep the center of total mass in an almost stationary position.

5 Linear Systems, Superposition

1. For a linear homogeneous differential equation, e.g. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$:
- if $x_1(t)$ and $x_2(t)$ are [homogenous] solutions to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$,

  then $a_1x_1(t) + a_2x_2(t)$ is a [homogenous] solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$.

2. For a linear equation of motion, e.g. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ with constant coefficients for $\ddot{x}$, $\dot{x}$, x:
- if $x_1(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_1(t)$,
- and if $x_2(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_2(t)$,

  then $a_1x_1(t) + a_2x_2(t)$ is a particular solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a_1f_2(t) + a_2f_2(t)$.

6 Response to a Periodic Excitation (Fourier Series)

Any periodic function $F(t)$ with period T could be represented by an infinite series of the form:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\, cos(n\omega_T t) + b_n\, sin(n\omega_T t)]$$

With $\omega_T = \frac{2\pi}{T}$, $a_0 = \frac{2}{T}\int_0^T F(t)dt$, $a_n = \frac{2}{T}\int_0^T F(t)\, cos(n\omega_T t)\, dt$ and $b_n = \frac{2}{T}\int_0^T F(t)\, sin(n\omega_T t)\, dt$. (Eq. 3-20 to 3.23). The superposition principle could be used to calculate the response to the periodic force by calculating the response to each Fourier term and adding the resulting displacements.

7 Response to impulse excitation, underdamped SDOF:

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t - \tau)$$
$$\Rightarrow x(t) = \hat{F}.h(t - \tau)$$

$$h(t-\tau) = \frac{1}{m\omega_d}.e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau) \qquad t \geq \tau \qquad \text{(eq. 3. 9)}$$

8 Response to arbitrary excitation

$$x(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t [F(\tau).e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)]d\tau = \frac{1}{m\omega_d}\int_0^t [F(t-\tau).e^{-\zeta\omega_n\tau}\,sin\,\omega_d\tau]d\tau \qquad (3.13)$$

9 Modal Analysis

(In this section, **boldface** is used to show matrices).

9.1 Modal Analysis of undamped free response

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$$

9.1.1 General mass matrix, by Cholesky decomposition

$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$ *(eq. 4.54)*

1. Calculate lower triangular matrix $\mathbf{L}$ such that $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$ **(see the footnote[2])**
2. Calculate $\mathbf{L}^{-1}$
3. Calculate the mass normalized stiffness matrix $\widetilde{\mathbf{K}} = \mathbf{L}^{-1}\mathbf{K}(\mathbf{L}^{-1})^{\mathrm{T}}$
4. Calculate the symmetric eigenvalue problem for $\widetilde{\mathbf{K}}$ to get ω_i^2 and Orthonormal eigenvectors v_i. Build $\mathbf{P}$ with these orthonormal eigenvectors:

$$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots]$$

[since $\widetilde{K}$ is a symmetric matrix its eigenvectors will be orthogonal to each other, i.e. $\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = 0$, But $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ should also be normalized, i.e. their norm $||\mathbf{v}_i||$, square root of sum of square of elements, should be 1. You can ensure this by updating the eigenvectors as $\mathbf{v}_{\mathrm{i}_{\mathrm{updated}}} = \mathbf{v}_{\mathrm{i}}/||\mathbf{v}_{\mathrm{i}}||$

Since the columns of $\mathbf{P}$ are orthonormal eigenvectors of $\widetilde{\mathbf{K}}$, then $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}_{n\times n}$ ($n \times n$ unity matrix) and $\mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{K}}\,\mathbf{P} = \mathbf{\Lambda}$.

and $\mathbf{\Lambda}$ is a diagonal matrix with square of natural frequencies for each modes shape as its main diagonal:

$$\mathbf{\Lambda} = \mathrm{diag}(\boldsymbol{\omega}_{\mathrm{i}}^2) = \begin{bmatrix} \omega_1^2 & 0 & & \cdots & & \\ 0 & \omega_2^2 & & & & \\ & & \ddots & & & \\ & & & \omega_{\mathrm{i}}^2 & & \\ & & & & \ddots & \\ & & & & & \omega_n^2 \end{bmatrix}$$

5. Calculate $\mathbf{S} = (\mathbf{L}^{-1})^{\mathrm{T}}\,\boldsymbol{P}$ and $\mathbf{S}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$
6. Calculate the modal initial condition vectors, $\mathbf{r}(\mathbf{0}) = \mathbf{S}^{-1}\mathbf{x}_0, \dot{\mathbf{r}}(\mathbf{0}) = \mathbf{S}^{-1}\dot{\mathbf{x}}_0$
7. Substitute $\mathbf{r}(\mathbf{0})$ and $\dot{\mathbf{r}}(\mathbf{0})$ into equations (4.66) and (4.67) to get the solution in modal coordinate $\mathbf{r}(\mathbf{t})$:

$$r_i\,(t) = \frac{\sqrt{\omega_i^2 r_{i,0}^2 + \dot{r}_{i,0}{}^2}}{\omega_i}\sin(\omega_i t + \mathrm{atan2}\,(\omega_i r_{i,0}\ ,\quad \dot{r}_{i,0})\ )\ , i = 1, 2, \dots$$

8. Multiply $\mathbf{r}(t)$ by $\boldsymbol{S}$ to get the solution $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(t)$

   Note that $\mathbf{S}$ is the matrix of mode shapes and $\mathbf{P}$ is the matrix of eigenvectors of $\widetilde{\mathbf{K}}$.

9.2 Modal Analysis of the Forced Response, with general mass matrix and damping

$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{F}(t)$ *(eq. 4.126)*

$\mathbf{B}\mathbf{F}(t)$ is used to shape application of various force functions on degrees of freedom.

1. Calculate lower triangular matrix $\mathbf{L}$ such that $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. For diagonal mass matrix see the footnote in the previous page.

If the damping matrix has specific conditions, e.g. it is a proportional to mass and stiffness matrices as:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

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[2] If you can easily calculate $\mathbf{M}^{\frac{1}{2}}$, (e.g. when you have a diagonal $\mathbf{M}$ matrix), then you can replace $\mathbf{L}$ by $\mathbf{M}^{\frac{1}{2}}$ in the remining of equations and $\mathbf{L}^{-1} = \mathbf{M}^{-1/2}$. With a diagonal $\mathbf{M}$ matrix directly take the square root of diagonal elements to calculate $\mathbf{L} = \mathbf{M}^{\frac{1}{2}}$. You can not do so if $\mathbf{M}$ was not a diagonal matrix.

The result of transformation $\tilde{\mathbf{C}} = \mathbf{L}^{-1}\mathbf{C}(\mathbf{L}^{-1})^{\mathrm{T}} = \alpha\mathrm{I} + \beta\tilde{\mathbf{K}}$ becomes diagonal if the matrix of eigenvectors of $\tilde{\mathbf{K}}$ are multiplied to it from the right $(\mathbf{P})$ and left $(\mathbf{P}^{\mathrm{T}})$ as follows:

$$\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$$

Replacing $\mathbf{x}(t)$ with $\mathbf{x}(t) = (\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{q}(t)$ in the differential equation (4.126) and multiplying $\mathbf{L}^{-1}$ from left results in:

$$\mathbf{I}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{q}}(t) + \tilde{\mathbf{K}}\mathbf{q}(t) = \mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$$ *(similar to eq. 4.128)*

Defining $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$, where $\mathbf{P}$ is the orthonormal eigenvector matrix of $\tilde{\mathbf{K}}$, [note that this results in $\mathbf{x}(t) = (\mathbf{L}^{-1})^{T}\ \mathbf{q}(t) = (\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{P}\,\mathbf{r}(t)$ and With $\mathbf{S} = (\mathbf{L}^{-1})^{T}\ \mathbf{P}$ and $\mathbf{S}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$ then $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(t)$ and $\mathbf{r}(t) = \mathbf{S}^{-1}\mathbf{x}(t)$]

replacing $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$ in *(eq. 4.128)* multiplying $\mathbf{P}^{\mathrm{T}}$from left to this equation results in:

$$\mathbf{I}_{n\times n}\ddot{\mathbf{r}}(t) + \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]\,\dot{\mathbf{r}}(t) + \mathbf{\Lambda}\mathbf{r}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$$ *(similar to eq. 4.129)*

In *above equation:*

- $\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$ and $\mathbf{\Lambda} = \mathbf{P}^{\mathrm{T}}\tilde{\mathbf{K}}\,\mathbf{P} = \mathrm{diag}(\boldsymbol{\omega}_{\mathrm{i}}^{2})$
- The vector $\mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ has elements $f_i(t)$ that will be linear combination of forces applied to the degrees of freedom.
- The modal initial conditions are calculated as $\mathbf{r}(\mathbf{0}) = \mathbf{S}^{-1}\mathbf{x}_0$ and $\dot{\mathbf{r}}(\mathbf{0}) = \mathbf{S}^{-1}\dot{\mathbf{x}}_0$
- The response for each mode (elements of $\mathbf{r}(t)$) could be calculated similar to the response of single degree of freedom systems with $f_i(t)$ excitation:

$$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_{\mathrm{i}}^{2}r_i(t) = f_i(t)$$

(e.g. if it is harmonic excitation by the same equations as in 2.1), . or by eq. 3.13.

The resulting $r_i(t)$s are assembled back in $\mathbf{r}(t)$.

- The response in natural coordinate system is obtained by $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(\mathrm{t})$

9.3 Physical, Mass Normalized and Modal Spaces

Eq.	Name	Mass Matrix	Damping Matrix	Stiffness Matrix	Matrix Transformation	State Vector	State Vect… ↓
$\mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{BF}(t)$	Physical Space	$\mathbf{M}$	$\mathbf{C}$	$\mathbf{K}$		$\mathbf{X}(t)$	
$\mathbf{I\ddot{q}} + \mathbf{\tilde{C}\dot{q}} + \mathbf{\tilde{K}q} = \mathbf{L}^{-1}\mathbf{BF}(t)$	Mass Normalized	$\mathbf{I}$	$\tilde{\mathbf{C}}$	$\tilde{\mathbf{K}}$	$\tilde{\mathbf{K}} = (\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T$	$\mathbf{q}(t)$	$\mathbf{q}(t) = \mathbf{L}^T\mathbf{X}(t$
$\ddot{\mathbf{r}} + \mathbf{diag}[2\zeta_i\omega_i]\dot{r} + \Lambda\mathbf{r} = \mathbf{P}^T\mathbf{L}^{-1}\mathbf{BF}(t)$ Decoupled differential equation*. Or for $i = 1$ *to* n: $\ddot{r}_i + 2\zeta_i\omega_i\dot{r}_i + \omega_i^2 r_i = f_i(t)$	Modal Space	$\mathbf{I}$	$[\mathrm{diag}(2\zeta_i\,\omega_i)]$ (*)	$\Lambda = [\mathrm{diag}(\omega_i^2)]$	$\Lambda = \mathbf{P}^T\tilde{\mathbf{K}}\mathbf{P} = \mathbf{P}^T(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T\mathbf{P} = \mathbf{S}^T\mathbf{K}\,\mathbf{S}$	$\mathbf{r}(t)$	$\mathbf{r}(t) = \mathbf{P}^T\mathbf{q}(t$ $= \mathbf{S}^{-1}\mathbf{X}(t)$

* ONLY IF $\mathbf{S}^T\mathbf{C}\,\mathbf{S}$ becomes a diagonal matrix $[\mathrm{diag}(2\zeta_i\,\omega_i)]$, e.g. when $\mathbf{C} = \alpha\mathbf{M} + \beta\,\mathbf{K}$, then $\mathbf{S}^T\mathbf{C}\,\mathbf{S} = \alpha\mathbf{I} + \beta\Lambda = [diag(2\zeta_i\,\omega_i)]$

Transformation Matrices:

	Description	Definition	Calculation in MATLAB
$\mathbf{L}$	Normalization of Mass Matrix Lower triangular Cholesky's matrix for $\mathbf{M}$	$\mathbf{M} = \mathbf{LL}^T$	`L = chol(M, 'lower');`
$\mathbf{P}$	Makes $\tilde{\mathbf{K}}$ diagonal	$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots]$ Matrix of Orthonormal Eigenvectors of $\tilde{\mathbf{K}}$ $\mathbf{P}^T\mathbf{P} = \mathbf{I}_{n\times n}$	`K_tilde = (L^(-1))*K*(L^(-1)` `[P, Lambda] = eig(K_tilde)`

$\mathbf{S}$	Matrix of Mode Shapes,
	 Moves from Modal Space to Physical Space
		$\mathbf{S} = (\mathbf{L}^{-1})^T\mathbf{P}$
		 Also $\mathbf{S}^{-1} = \mathbf{P}^T\mathbf{L}^T$ and
		 $\mathbf{S}^T = \mathbf{P}^T(\mathbf{L}^{-1})$
		 (in general, $\mathbf{S}^T \neq \mathbf{S}^{-1}$)
			`S = (L^(-1))' * P`
			 `% Or`
			 `S = (L')\P`

Physical, Mass Normalized and Modal Spaces with [SI units] (for translational mass systems):

Eq.	Name	Mass Matrix	Damping Matrix	Stiffness Matrix	Matrix Transformation	State Vector	State Vector T	
							↓	
$\mathbf{M}[\mathrm{kg}]\ddot{\mathbf{x}}\left[\frac{\mathrm{m}}{s^2}\right]+\mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right]\dot{\mathbf{x}}\left[\frac{\mathrm{m}}{s}\right]+\mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]\mathbf{x}[\mathrm{m}]=\mathbf{BF}(t)\ [\mathrm{N}]$	Physical Space	$\mathbf{M}[\mathrm{kg}]$	$\mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right]$	$\mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$		$\mathbf{X}(t)[m]$		$\mathbf{X}($ $=$
$\mathbf{I}\ddot{\mathbf{q}}\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]+\tilde{\mathbf{C}}\dot{\mathbf{q}}+\tilde{\mathbf{K}}\mathbf{q}=\mathbf{L}^{-1}\mathbf{BF}(t)\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]$	Mass Normalized	$\mathbf{I}[-]$	$\tilde{\mathbf{C}}\left[\frac{\mathrm{N.s}}{\mathrm{kg.m}}=\frac{1}{\mathrm{s}}\right]$	$\tilde{\mathbf{K}}\left[\frac{\mathrm{N}}{\mathrm{kg.m}}=\frac{1}{\mathrm{s}^2}\right]$	$\tilde{\mathbf{K}}=(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T$	$\mathbf{q}(t)[\mathrm{m}\sqrt{\mathrm{kg}}]$	$\mathbf{q}(t)=\mathbf{L}^{\mathrm{T}}\mathbf{X}(t)$	$\mathbf{q}($
$\ddot{\mathbf{r}}\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]+\mathbf{diag}[2\zeta_i\omega_i]\dot{\mathbf{r}}+\boldsymbol{\Lambda}\mathbf{r}=\mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{BF}(t)\left[\frac{\mathrm{m}\sqrt{\mathrm{kg}}}{\mathrm{s}^2}\right]$ Decoupled differential equation*, for $i\ =\ 1\ to\ n$: $\ddot{r}_i+2\zeta_i\omega_i\dot{r}_i+\omega_i^2 r_i=f_i(t)$	Modal Space	$\mathbf{I}[-]$	$[\mathrm{diag}(2\zeta_i\ \omega_i)]=\left[\frac{1}{\mathrm{s}}\right]$ (*)	$\boldsymbol{\Lambda}\left[\frac{1}{\mathrm{s}^2}\right]=[\mathrm{diag}(\omega_i^2)]$	$\boldsymbol{\Lambda}=\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{K}}\mathbf{P}=\mathbf{P}^{\mathrm{T}}(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T\mathbf{P}=\mathbf{S}^{\mathrm{T}}\mathbf{K}\,\mathbf{S}$	$\mathbf{r}(t)[\mathrm{m}\sqrt{\mathrm{kg}}]$	$\mathbf{r}(t)=\mathbf{P}^{\mathrm{T}}\mathbf{q}(t)=\mathbf{S}^{-1}\mathbf{X}(t)$	

* ONLY IF $\mathbf{S}^{\mathrm{T}}\mathbf{C}\,\mathbf{S}$ becomes a diagonal matrix $[\mathrm{diag}(2\zeta_i\ \omega_i)]$, e.g. when $\mathbf{C}\left[\frac{\mathrm{N.s}}{\mathrm{m}}\right]=\alpha\left[\frac{1}{\mathrm{s}}\right]\mathbf{M}[\mathrm{kg}]+\beta[\mathrm{s}]\,\mathbf{K}\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$, then $\mathbf{S}^{\mathrm{T}}\mathbf{C}\,\mathbf{S}=\alpha\mathbf{I}+\beta\boldsymbol{\Lambda}=[diag(2\zeta_i\ \omega_i)$

Transformation Matrices:

	Description	Definition	Calculation in MATLAB
$\mathbf{L}[\sqrt{\mathrm{kg}}]$	Normalization of Mass Matrix Lower triangular Cholesky's matrix for $\mathbf{M}$	$\mathbf{M}=\mathbf{LL}^{\mathrm{T}}$	`L = chol(M, 'lower');`
$\mathbf{P}[-]$	Makes $\tilde{\mathbf{K}}$ diagonal	$\mathbf{P}=[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\dots]$ Matrix of Orthonormal Eigenvectors of $\tilde{\mathbf{K}}$ $\mathbf{P}^{\mathrm{T}}\mathbf{P}=\mathbf{I}$	`K_tilde = (L^(-1))*K*(L^(` `[P, Lambda] = eig(K_tilde`
$\mathbf{S}\left[\frac{1}{\sqrt{\mathrm{kg}}}\right]$	Matrix of Mode Shapes, Moves from Modal Space to Physical Space	$\mathbf{S}=(\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{P}$ Also $\mathbf{S}^{-1}[\sqrt{\mathrm{kg}}]=\mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$ and $\mathbf{S}^{\mathrm{T}}\left[\frac{1}{\sqrt{\mathrm{kg}}}\right]=\mathbf{P}^{\mathrm{T}}(\mathbf{L}^{-1})$ (in general, $\mathbf{S}^{\mathrm{T}}\neq\mathbf{S}^{-1}$)	`S = (L^(-1))' * P` `% Or` `S = (L')\P`

10 Power/Logarithm

$e^a = \mathrm{b} \Leftrightarrow a = \ln(b)$

11 Matrix Identities

If k is a scalar then $k\mathbf{A} = \mathbf{A}k$

Matrix to vector multiplication:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{v} = \begin{bmatrix} e \\ g \end{bmatrix}$$

$$\Rightarrow \mathbf{Av} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$$

Matrix to matrix multiplication (for 2x2 matrices):

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\Rightarrow \mathbf{AB} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Compatibility: For matrix multiplication to be defined: $\mathbf{A}_{m\times \mathrm{n}}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$

Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Distributivity: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

Identity Matrix: $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$.

Not Commutative: in general: $\mathbf{AB} \neq \mathbf{BA}$

Determinant of multiplication:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Transpose of a 2x2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Transpose of product: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Determinant and Inverse of a 2x2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(\mathbf{A}) = ad - bc$$

Inverse of a 2x2 matrix:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

12 Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

13 Trigonometric Identities

Pythagorean identity: $sin^2\,\theta + cos^2\,\theta = 1$

Angle Sum:

$$\sin(\alpha + \beta) = \sin\alpha\,\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\ \ sin\,\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product-to-sum

$$\cos\theta\cos\varphi = \frac{\cos(\theta - \varphi) + \cos(\theta + \varphi)}{2}$$

$$\sin\theta\sin\varphi = \frac{\cos(\theta - \varphi) - \cos(\theta + \varphi)}{2}$$

$$\sin\theta\cos\varphi = \frac{\sin(\theta + \varphi) + \sin(\theta - \varphi)}{2}$$

$$\cos\theta\sin\varphi = \frac{\sin(\theta + \varphi) - \sin(\theta - \varphi)}{2}$$

Sum-to-product

$$\sin\theta \pm \sin\varphi = 2\sin\left(\frac{\theta \pm \varphi}{2}\right)\cos\left(\frac{\theta \mp \varphi}{2}\right)$$

$$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta + \varphi}{2}\right)\sin\left(\frac{\theta - \varphi}{2}\right)$$

$$\cos\theta + \cos\varphi = 2\,cos\left(\frac{\theta + \varphi}{2}\right)\cos\left(\frac{\theta - \varphi}{2}\right)$$

$$\tan\theta \pm \tan\varphi = \frac{\sin(\theta \pm \varphi)}{\cos\theta\cos\varphi}$$

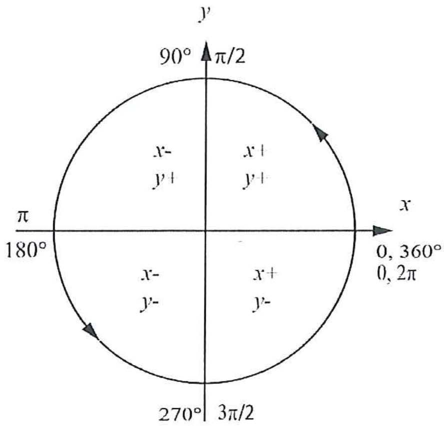
14 Quadratic equation

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Reduced form (a = 1):

$$x^2 + px + q = 0 \Rightarrow x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

15 Four Quadrant Arctangent Function, $\mathrm{atan2}(y, x)$



$$\mathrm{atan2}(y,x) = \begin{cases} \mathrm{atan}\left(\frac{y}{x}\right) & x > 0 \\ \mathrm{atan}\left(\frac{y}{x}\right) + \pi & y \geq 0; x < 0 \\ \mathrm{atan}\left(\frac{y}{x}\right) - \pi & y < 0; x < 0 \\ \frac{\pi}{2} & y > 0; x = 0 \\ -\frac{\pi}{2} & y < 0; x = 0 \\ \text{undefined} & y = x = 0 \end{cases}$$

In MATLAB and many other software, the correct form is atan2(y,x), **but** in Excel, you should enter ATAN2(x;y) to get the correct answer.