UNIVERSITY
OF SKÖVDE

School of Engineering Science

WRITTEN EXAMINATION

Course: Mekanik IV / Mechanics IV

Sub-course

Course code: MT355G

Credits for written examination: 3 hp

Date: 2023-11-17

Examination time: 08:15-13:30

Examination responsible: Mahdi Eynian

Teachers concerned: Mahdi Eynian, Daniel Svensson

Aid at the exam/appendices

- Inman D. J. (2014). Engineering Vibrations. (4th ed) Essex England: Pearson. ISBN 9780273768449

- Råde, L, Westergren, B. (1990). *Beta – Mathematics Handbook*. Lund: Studentlitteratur.
Or a similar handbook

- Sundström, B. (red.) (2010). *Handbook of Solid Mechanics*. Stockholm: Department of Solid Mechanics, KTH. ISBN 9789197286046.
Or the Swedish version

- Sundström, B. (1999). *Handbok och formelsamling i hållfasthetslära*. Tekniska högskolan Stockholm: Institution för hållfasthetslära.

- An approved calculator according to "Allmänna riktlinjer gällande utbildning på Institutionen for ingenjörsvetenskap":
  - Casio Teknikräknare FX-82 all variants
  - Texas Instruments Tl-30 all variants
  - Texas Instruments Tl-82, Tl-83, Tl-84
  - Casio FX-7400Gii, Fx-9750GII

- An English-Swedish-English ordbok or English-Spanish-English dictionary.

No added notes are allowed in the texts used during the examination.

Other

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Instructions

- ☐ Take a new sheet of paper for each teacher.
- ☒ Take a new sheet of paper when starting a new question.
- ☒ Write only on one side of the paper.
- ☒ Write your name and personal ID No. on all pages you hand in.
- ☒ Use page numbering.
- ☒ Don´t use a red pen.
- ☒ Mark answered questions with a cross on the cover sheet.

Grade points

Examination results should be made public within 18 working days

Good luck!

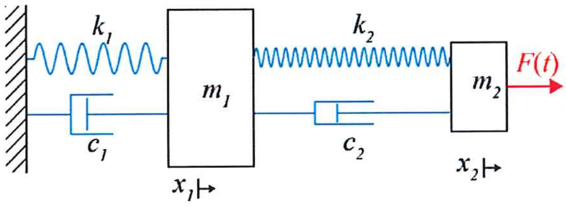
Total number of pages

Written exam MT355G- Mechanics IV 2023-11-17

Question 1 (10p)

The following dynamic parameters are given for the system below:

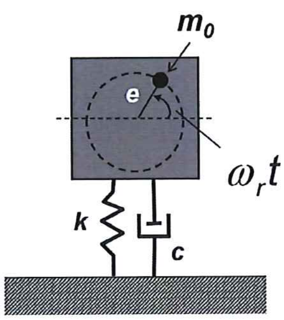
$$k_1 = 3000\frac{\text{N}}{\text{m}}, k_2 = 1000\frac{\text{N}}{\text{m}}, m_1 = 16\text{ kg}, m_2 = 4\text{ kg}, c_1 = 30\frac{\text{Ns}}{\text{m}}, c_2 = 10\frac{\text{Ns}}{\text{m}}$$



a) Obtain the mode shapes and natural frequencies using modal analysis approach. (5p)
b) Obtain the response of the system if the excitation force on m_2 was $F(t) = 500\,\delta(t-\tau)$, and $\tau = 1$, assume zero initial velocity and position for the masses. (5p)

Question 2 (5 p)

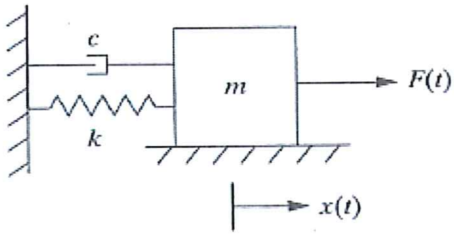
In the folowing system, the rotation speed in rad/s is numerically equal to the natural frequency of the mass-spring system, i.e. $\omega_r = \omega_n = 40$ rad/s. The machine is modelled as having a stiffness 4000 N/m, the eccentric mass is 0.01 kg, at a radius of e = 0.15 m.



a) Design a damper (that is, chose a value of c) such that the maximum deflection at steady state is 0.002 m. (2p)
b) find the allowable driving frequencies (below and above the natural frequency) that create maximum allowable deflection at steady state below 0.001 m (with the damper value chosen above). (3p)

General Case:

$m\ddot{x} + c\dot{x} + kx = F(t)$ Newton's 2nd Law (NSL)

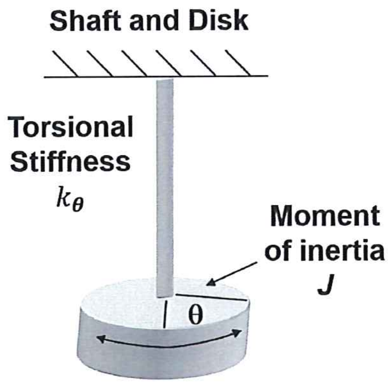


with $\zeta = \frac{c}{2\sqrt{k.m}}, \omega_n = \sqrt{k/m}, f(t) = F(t)/m$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ Mass normalized form

initial conditions: $x(0) = x_0$ and $\dot{x}(0) = v_0$

Dynamics of rotating systems



$$J[\mathrm{kg \cdot m^2}]\,\ddot{\theta} + c_\theta\left[\frac{\mathrm{N \cdot m}}{\mathrm{s}}\right]\dot{\theta} + k_\theta[\mathrm{N \cdot m}]\theta = T(t)[\mathrm{N \cdot m}]$$

with $\zeta = \frac{c_\theta}{2\sqrt{k_\theta . J}}, \omega_n = \sqrt{\frac{k_\theta}{J}}, f(t) = T(t)/J$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)$$

1 Free Vibration ($F(t) = 0$)

1.1 Undamped Case ($c = 0$)

$$\ddot{x} + \omega_n^2 x = 0$$

solution:

$x(t) = a.\sin(\omega_n t) + b.\cos(\omega_n t)$	$a = \frac{v_0}{\omega_n}$	$b = x_0$
	Or	
$x(t) = A.\sin(\omega_n t + \phi)$	$A = \frac{\sqrt{x_0^2\omega_n^2 + v_0^2}}{\omega_n}$	$\phi = \arctan\left(\frac{x_0\omega_n}{v_0}\right)$

1.2 Damped Case (With Viscous Damping, $c \neq 0$)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \zeta = \frac{c}{2\sqrt{k.m}} \neq 0$$

1.2.1 Underdamped case ($0 < \zeta < 1$)

solution:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$x(t) = e^{-\zeta\omega_n t}[a.\sin(\omega_d t) + b.\cos(\omega_d t)]$	$a = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$	$b = x_0$
	Or	
$x(t) = Ae^{-\zeta\omega_n t}.\sin(\omega_d t + \phi)$	$A = \frac{\sqrt{\omega_d^2 x_0^2 + (v_0^2 + \zeta\omega_n x_0)^2}}{\omega_d}$	$\phi = \arctan\left(\frac{\omega_d x_0}{v_0 + \zeta\omega_n x_0}\right)$

1.2.2 Critically damped case ($\zeta = 1$)

$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$	$a_1 = x_0$	$a_2 = v_0 + \omega_n x_0$
Eq. 1.45	Eq. 1.46	Eq. 1.46

1.2.3 Overdamped case ($\zeta > 1$)

$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-(\omega_n\sqrt{\zeta^2-1})t} + a_2 e^{+(\omega_n\sqrt{\zeta^2-1})t}\right)$	$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$	$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$
Eq. 1.41	Eq. 1.42	Eq. 1.43

2 Forced Vibration ($F(t) \neq 0$)

The general response will be the sum of the free vibration response added to the particular solution, $x_p(t)$. The sum should satisfy the initial conditions (coefficients such as a,b for the free response should be adjusted and will not always follow the same relationships for the free vibration). In damped systems, after a while, the response from the initial conditions will die out and the system's vibration will be dominated by the particular response (solution).

2.1 Harmonic excitation $F(t) = F_0\cos(\omega t)$ or $f(t) = \frac{F(t)}{m} = f_0\cos(\omega t)$

($f_0 = \frac{F_0}{m}$, note that the SI unit for f_0 is $\left[\frac{\text{N}}{\text{kg}}\right] = \left[\frac{\text{m}}{\text{s}^2}\right]$)

2.1.1 Undamped Case (c = 0, $\zeta = 0, r \neq 1$)

Differential equation: $\ddot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution: $x_p(t) = \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$

Total solution with IC: $x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos\omega_n t + \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$ (eq. 2.11)

Zero initial conditions in this case will lead to beating, with amplitude $\left|\frac{2f_0}{\omega_n^2-\omega^2}\right|$ and beat frequency of $\omega_{beat} = |\omega_n - \omega|$

$$x(t) = \frac{2f_0}{\omega_n^2-\omega^2}\sin\left(\frac{\omega_n-\omega}{2}\right)\sin\left(\frac{\omega_n+\omega}{2}\right) \quad \text{(eq. 2.13)}$$

2.1.2 Resonance at undamped case ($c = 0, \omega = \omega_n$) or ($\zeta = 0, r = 1$)

$x(t) = A_1\sin\omega t + A_2\cos\omega t + \frac{f_0}{2\omega}t\sin\omega t$ (eq. 2.17), A_1, A_2 depend on initial conditions.

2.1.3 Damped Case

Differential equation: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos(\omega t)$

Particular solution: $x_p(t) = X\cos(\omega.t - \theta)$

$$X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega.\omega_n)^2}} = \frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}}.\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

and $\frac{\boldsymbol{f_0}}{\boldsymbol{\omega_n^2}} = \frac{F_0}{m\omega_n^2} = \frac{F_0}{k}$ (i.e. displacement of the spring if F_0 was applied statically)

$$\theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right) = \arctan\left(\frac{2\,\zeta\,r}{1-r^2}\right)$$

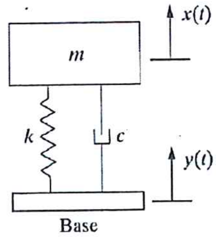
2.1.3.1 *Resonance (for $0 \leq \zeta \leq \frac{1}{\sqrt{2}}$)*

$$\frac{\omega_{peak}}{\omega_n} = r_{peak} = \sqrt{1-2\zeta^2}$$

$$X_{peak} = \frac{f_0}{\omega_n^2}.\frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{2k\zeta\sqrt{1-\zeta^2}} \overbrace{\cong \frac{F_0}{2k\zeta}}^{\text{if } \zeta \ll 1}$$

3 Base Excitation

3.1 Harmonic Excitation



$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0, \quad y = Y\sin(\omega t)$ from NSL

Standard form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega Y \times \cos(\omega t) + \omega_n^2 Y \times \sin(\omega t)$$

Particular solution:

$$x_p(t) = X\,.\sin(\omega.t - \psi)$$

$$X = Y.\sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

Or

$$X = Y.\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

$$\psi = \arctan\left[\frac{mc\omega^3}{k(k - m\omega^2) + (c.\omega)^2}\right] = \arctan\left[\frac{2\zeta r^3}{(1 - r^2) + (2\zeta r)^2}\right]$$

$\left|\frac{X}{Y}\right| =. \sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **displacement transmissibility** and reaches to its maximum (resonance) very close to $r = 1$. This ratio reduces and reaches 0 as the r increases (i.e. when base vibration frequency increases to values much higher than the natural frequency, the mass remains almost still. In other words, you cannot oscillate an object at frequencies much higher than the natural frequency that is created between that object and its base). (Fig. 2.14)

3.1.1 Transmitted Force

$$F(t) = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x} = -m\omega^2 X.\sin(\omega.t - \psi)$$

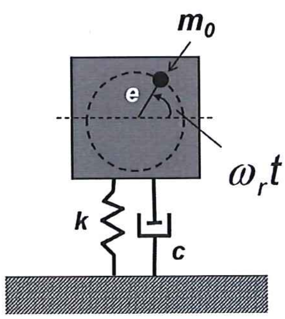
$$\left|\frac{F}{Y}\right| = m\omega^2\left|\frac{X}{Y}\right| = (k.r^2) \times \sqrt{\frac{1 + (2.\zeta.r)^2}{(1 - r^2)^2 + (2.\zeta.r)^2}}.$$

$\left|\frac{F}{kY}\right| = r^2 \times \sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **force transmissibility ratio**. This function also has a local peak very close to $r = 1$ With non-zero damping ratio it keeps increasing as the r ratio increases. (Fig. 2.15 in the book).

4 Rotating Unbalance

NSL: $$m\ddot{x} + c\dot{x} + kx = m_0 e\omega_r^2 \sin(\omega_r t) = F_0 \sin(\omega_r t)$$

(m is the total mass of the machine, including the unbalance mass. $m_0$is the unbalanced mass, that rotates with eccentricity e and angular velocity of ω_r).



Particular solution: $$x_p(t) = X\sin(\omega_r t - \theta)$$

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega_r^2)^2 + (2\zeta\omega_r\omega_n)^2}} = e.\frac{m_0}{m}.\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega_r}{\omega_n}$$

$$\theta = \arctan\left(\frac{2\zeta\omega_n\omega_r}{\omega_n^2 - \omega_r^2}\right) = \arctan\left(\frac{2\,\zeta\,r}{1 - r^2}\right)$$

5 Response to impulse excitation, underdamped SDOF:

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t - \tau)$$
$$\Rightarrow x(t) = \hat{F}.h(t - \tau)$$

$$h(t - \tau) = \frac{1}{m\omega_d} . e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau) \qquad t \geq \tau \qquad \text{(eq. 3. 9)}$$

6 Response to arbitrary excitation

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t [F(\tau). e^{\zeta\omega_n \tau} \sin \omega_d(t - \tau)] d\tau = \frac{1}{m\omega_d} \int_0^t [F(t - \tau). e^{-\zeta\omega_n \tau} \sin \omega_d \tau] d\tau \qquad (3.13)$$

7 Modal Analysis

(In this section, **boldface** is used to show matrices).

7.1 Modal Analysis of undamped free response

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$$

7.1.1 General mass matrix, by Cholesky decomposition

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \qquad \textit{(eq. 4.54)}$$

1. Calculate lower triangular matrix $\mathbf{L}$ such that $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$ **(see the footnote[1])**
2. Calculate $\mathbf{L}^{-1}$
3. Calculate the mass normalized stiffness matrix $\widetilde{\mathbf{K}} = \mathbf{L}^{-1}\mathbf{K}(\mathbf{L}^{-1})^{\mathrm{T}}$
4. Calculate the symmetric eigenvalue problem for $\widetilde{\mathbf{K}}$ to get ω_i^2 and Orthonormal eigenvectors $\boldsymbol{v}_i$. Build $\mathbf{P}$ with these orthonormal eigenvectors:

$$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots]$$

[since $\widetilde{K}$ is a symmetric matrix its eigenvectors will be orthogonal to each other, i.e. $\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = 0$, But $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ should also be normalized, i.e. their norm $\|\mathbf{v}_i\|$, square root of sum of square of elements, should be 1. You can ensure this by updating the eigenvectors as $\mathbf{v}_{i_{\mathrm{updated}}} = \mathbf{v}_i/\sqrt{\|\mathbf{v}_i\|}$

Since the columns of $\mathbf{P}$ are orthonormal eigenvectors of $\widetilde{\mathbf{K}}$, then $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}_{n\times n}$ ($n \times n$ unity matrix) and $\mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{K}}\,\mathbf{P} = \boldsymbol{\Lambda}$.

and $\boldsymbol{\Lambda}$ is a diagonal matrix with square of natural frequencies for each modes shape as its main diagonal:

$$\boldsymbol{\Lambda} = \mathrm{diag}(\boldsymbol{\omega}_i^2) = \begin{bmatrix} \omega_1^2 & 0 & \cdots & & \\ 0 & \omega_2^2 & & & \\ & & \ddots & & \\ & & & \omega_i^2 & \\ & & & & \ddots \\ & & & & & \omega_n^2 \end{bmatrix}$$

5. Calculate $\mathbf{S} = (\mathbf{L}^{-1})^{\mathrm{T}} \boldsymbol{P}$ and $\mathbf{S}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$
6. Calculate the modal initial condition vectors, $\mathbf{r}(\mathbf{0}) = \mathbf{S}^{-1}\mathbf{x}_0, \dot{\mathbf{r}}(\mathbf{0}) = \mathbf{S}^{-1}\dot{\mathbf{x}}_0$
7. Substitute , $\mathbf{r}(\mathbf{0})$ and $\dot{\mathbf{r}}(\mathbf{0})$ into equations (4.66) and (4.67) to get the solution in modal coordinate $\mathbf{r(t)}$:

$$r_i\,(t) = \frac{\sqrt{\omega_i^2 r_{i,0}^2 + \dot{r}_{i,0}{}^2}}{\omega_i} \sin\left(\omega_i t + \arctan\frac{\omega_i r_{i,0}}{\dot{r}_{i,0}}\right), i = 1, 2, \dots$$

8. Multiply , $\mathbf{r}(t)$ by $\boldsymbol{S}$ to get the solution $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(t)$

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[1] If you can easily calculate $\mathbf{M}^{\frac{1}{2}}$, (e.g. when you have a diagonal $\mathbf{M}$ matrix), then you can replace $\mathbf{L}$ by $\mathbf{M}^{\frac{1}{2}}$ in the remining of equations and $\mathbf{L}^{-1} = \mathbf{M}^{-1/2}$. With a diagonal $\mathbf{M}$ matrix directly take the square root of diagonal elements to calculate $\mathbf{L} = \mathbf{M}^{\frac{1}{2}}$. You can not do so if $\mathbf{M}$ was not a diagonal matrix.

Note that $\mathbf{S}$ is the matrix of mode shapes and $\mathbf{P}$ is the matrix of eigenvectors of $\widetilde{\mathbf{K}}$.

7.2 Modal Analysis of the Forced Response, with general mass matrix and damping

$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{F}(t)$ *(eq. 4.126)*

$\mathbf{B}\mathbf{F}(t)$ is used to shape application of various force functions on degrees of freedom.

1. Calculate lower triangular matrix $\mathbf{L}$ such that $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. For diagonal mass matrix see the footnote in the previous page.

If the damping matrix has specific conditions, e.g. it is a proportional to mass and stiffness matrices as:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

The result of transformation $\tilde{\mathbf{C}} = \mathbf{L}^{-1}\mathbf{C}(\mathbf{L}^{-1})^{\mathrm{T}} = \alpha\mathrm{I} + \beta\widetilde{\mathbf{K}}$ becomes diagonal if the matrix of eigenvectors of $\widetilde{\mathbf{K}}$ are multiplied to it from the right $(\mathbf{P})$ and left $(\mathbf{P}^{\mathrm{T}})$ as follows:

$$\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$$

Replacing $\mathbf{x}(t)$ with $\mathbf{x}(t) = (\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{q}(t)$ in the differential equation (4.126) and multiplying $\mathbf{L}^{-1}$ from left results in:

$\mathbf{I}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{q}}(t) + \widetilde{\mathbf{K}}\mathbf{q}(t) = \mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ *(similar to eq. 4.128)*

Defining $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$, where $\mathbf{P}$ is the orthonormal eigenvector matrix of $\widetilde{\mathbf{K}}$, [note that this results in $\mathbf{x}(t) = (\mathbf{L}^{-1})^{T}\ \mathbf{q}(t) = (\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{P}\,\mathbf{r}(t)$ and With $\mathbf{S} = (\mathbf{L}^{-1})^{T}\ \mathbf{P}$ and $\mathbf{S}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$ then $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(t)$ and $\mathbf{r}(t) = \mathbf{S}^{-1}\mathbf{x}(t)$]

replacing $\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t)$ in *(eq. 4.128)* multiplying $\mathbf{P}^{\mathrm{T}}$from left to this equation results in:

$\mathbf{I}_{n\times n}\ddot{\mathbf{r}}(t) + \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]\ \dot{\mathbf{r}}(t) + \mathbf{\Lambda}\mathbf{r}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ *(similar to eq. 4.129)*

In *above equation:*

- $\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}\mathbf{P} = \mathbf{diag}[2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}]$ and $\mathbf{\Lambda} = \mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{K}}\,\mathbf{P} = \mathrm{diag}(\boldsymbol{\omega}_{\mathrm{i}}^{2})$
- The vector $\mathbf{P}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{F}(t)$ has elements $f_i(t)$ that will be linear combination of forces applied to the degrees of freedom.
- The modal initial conditions are calculated as $\mathbf{r}(\mathbf{0}) = \mathbf{S}^{-1}\mathbf{x}_0$ and $\dot{\mathbf{r}}(\mathbf{0}) = \mathbf{S}^{-1}\dot{\mathbf{x}}_0$
- The response for each mode (elements of $\mathbf{r}(t)$) could be calculated similar to the response of single degree of freedom systems with $f_i(t)$ excitation:

$$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_{\mathrm{i}}^{2}r_i(t) = f_i(t)$$

(e.g. if it is harmonic excitation by the same equations as in 2.1), . or by eq. 3.13.

The resulting $r_i(t)$s are assembled back in $\mathbf{r}(t)$.

- The response in natural coordinate system is obtained by $\mathbf{x}(t) = \mathbf{S}\,\mathbf{r}(\mathrm{t})$

7.3 Physical, Mass Normalized and Modal Spaces

Eq.	Name	Mass Matrix	Damping Matrix	Stiffness Matrix	Matrix Transformation	State Vector	State Vector Transformation		Force Vector
							↓	↑	
$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{BF}(t)$	Physical Space	$\mathbf{M}$	$\mathbf{C}$	$\mathbf{K}$		$\mathbf{X}(t)$		$\mathbf{X}(t) = (\mathbf{L}^{-1})^T\mathbf{q}(t) = \mathbf{S}\,\mathbf{r}(t)$	$\mathbf{BF}(t)$
$\mathbf{I}\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \mathbf{L}^{-1}\mathbf{BF}(t)$	Mass Normalized	$\mathbf{I}$	$\tilde{\mathbf{C}}$	$\tilde{\mathbf{K}}$	$\tilde{\mathbf{K}} = (\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T$	$\mathbf{q}(t)$	$\mathbf{q}(t) = \mathbf{L}^T\mathbf{X}(t)$	$\mathbf{q}(t) = \mathbf{Pr}(t)$	$\mathbf{L}^{-1}\mathbf{BF}(t)$
$\ddot{\mathbf{r}} + \mathbf{diag}[2\zeta_i\omega_i]\dot{\mathbf{r}} + \Lambda\mathbf{r} = \mathbf{P}^T\mathbf{L}^{-1}\mathbf{BF}(t)$ Decoupled differential equation.	Modal Space	$\mathbf{I}$	$[\mathrm{diag}(2\zeta_i\,\omega_i)]$ (*)	$\Lambda = [\mathrm{diag}(\omega_i^2)]$	$\Lambda = \mathbf{P}^T\tilde{\mathbf{K}}\mathbf{P} = \mathbf{P}^T(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T\mathbf{P} = \mathbf{S}^T\mathbf{K}\,\mathbf{S}$	$\mathbf{r}(t)$	$\mathbf{r}(t) = \mathbf{P}^T\mathbf{q}(t) = \mathbf{S}^{-1}\mathbf{X}(t)$		$\mathbf{f}(t) = \mathbf{P}^T\mathbf{L}^{-1}\mathbf{BF}(t) = \mathbf{S}^T\mathbf{BF}(t)$

* Possible if $\mathbf{S}^T\mathrm{C}\,\mathbf{S}$ becomes a diagonal matrix $[\mathrm{diag}(2\zeta_i\,\omega_i)]$, e.g. when $\mathbf{C} = \alpha\mathbf{M} + \beta\,\mathbf{K}$, then $\mathbf{S}^T\mathrm{C}\,\mathbf{S} = \alpha\mathrm{I} + \beta\Lambda = [diag(2\zeta_i\,\omega_i)]$

Transformation Matrices:

	Description	Definition	Calculation in MATLAB	With Diagonal M
$\mathbf{L}$	Normalization of Mass Matrix Lower triangular Cholesky's matrix for $\mathbf{M}$	$\mathbf{M} = \mathbf{L}\mathbf{L}^T$	`L = chol(M, 'lower');`	$\mathbf{L} = \mathbf{M}^{1/2}$
$\mathbf{P}$	Makes $\tilde{\mathbf{K}}$ diagonal	$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots]$ Matrix of Orthonormal Eigenvectors of $\tilde{\mathbf{K}}$	`K_tilde = (L^(-1))*K*(L^(-1))';` `[P, Lambda] = eig(K_tilde)`	

$\mathbf{S}$	Matrix of Mode Shapes,
	Moves from Modal Space to Physical Space
		$\mathbf{S} = (\mathbf{L}^{-1})^T\mathbf{P}$
		Also $\mathbf{S}^{-1} = \mathbf{P}^T\mathbf{L}^T$ and
		$\mathbf{S}^T = \mathbf{P}^T(\mathbf{L}^{-1})$
		(in general, $\mathbf{S}^T \neq \mathbf{S}^{-1}$)
			`S = (L^(-1))' * P`
			`% Or`
			`S = (L')\P`	

8 Power/Logarithm:

$$e^a = \mathrm{b} \Leftrightarrow a = \ln(b)$$

9 Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

10 Summary of Trigonometric Identities

Pythagorean trigonometric identity:

$$sin^2\,\theta + cos^2\,\theta = 1$$

10.1 Angle Sum:

$$\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\ \ sin\,\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

10.2 Product-to-sum

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$

$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$

$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$

$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

10.3 Sum-to-product

$$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$$

$$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$\cos\theta+\cos\varphi = 2\,cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$\tan\theta\pm\tan\varphi = \frac{\sin(\theta\pm\varphi)}{\cos\theta\cos\varphi}$$

11 Summary of Matrix Identities

Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Distributivity: $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

Identity Matrix: $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$.

Not Commutative: in general, $\mathbf{AB} \neq \mathbf{BA}$

Scalar Multiplication: If k is a scalar then $k\mathbf{A} = \mathbf{A}k$

Determinant of multiplication:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Transpose of product: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Compatibility: For matrix multiplication to be defined: $\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$

Determinant and Inverse of a 2x2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(\mathbf{A}) = ad - bc$$

and

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$