UNIVERSITY OF SKÖVDE

School of Engineering Science

# WRITTEN EXAMINATION

Course: Mekanik IV / Mechanics IV

Sub-course

Course code: MT355G

Credits for written examination: 3 hp

Date: 2023-12-18

Examination time: 08:15-13:30

Examination responsible: Mahdi Eynian

Teachers concerned: Mahdi Eynian, Daniel Svensson

Aid at the exam/appendices

- Inman D. J. (2014). Engineering Vibrations. (4th ed) Essex England: Pearson. ISBN 9780273768449

- Råde, L, Westergren, B. (1990). *Beta – Mathematics Handbook.* Lund: Studentlitteratur.
Or a similar handbook

- Sundström, B. (red.) (2010). *Handbook of Solid Mechanics.* Stockholm: Department of Solid Mechanics, KTH. ISBN 9789197286046.
Or the Swedish version

- Sundström, B. (1999). *Handbok och formelsamling i hållfasthetslära.* Tekniska högskolan Stockholm: Institution för hållfasthetslära.

- An approved calculator according to "Allmänna riktlinjer gällande utbildning på Institutionen for ingenjörsvetenskap":
  - Casio Teknikräknare FX-82 all variants
  - Texas Instruments Tl-30 all variants
  - Texas Instruments Tl-82, Tl-83, Tl-84
  - Casio FX-7400Gii, Fx-9750GII

- An English-Swedish-English ordbok or English-Spanish-English dictionary.

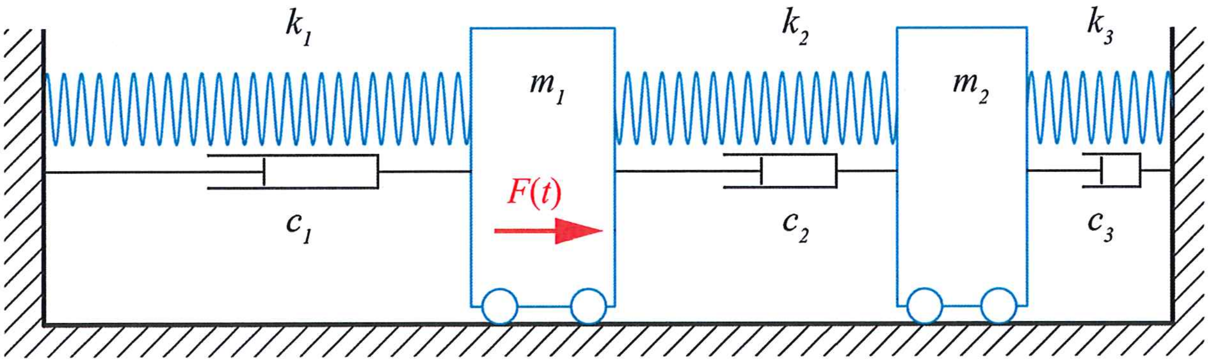
*No added notes are allowed in the texts used during the examination.*

Other

**Question 1 (10p)**

a) The following parameters are given for the system shown in the figure below:

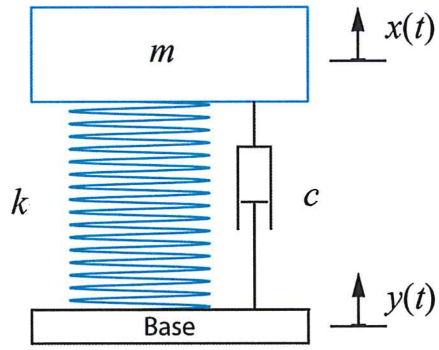
$$k_1 = 3000\frac{\mathrm{N}}{\mathrm{m}}, k_2 = 1000\frac{\mathrm{N}}{\mathrm{m}}, k_3 = 300\frac{\mathrm{N}}{\mathrm{m}}, m_1 = 16\ \mathrm{kg}, m_2 = 4\ \mathrm{kg},$$
$$c_1 = 31.6\frac{\mathrm{Ns}}{\mathrm{m}}, c_2 = 10\frac{\mathrm{Ns}}{\mathrm{m}}, c_3 = 3.4\ \frac{\mathrm{Ns}}{\mathrm{m}}$$




a) Obtain the mode shapes and natural frequencies using the modal analysis approach. (5p)

b) Obtain **the particular response** (in other words, steady state**,** ignoring the transient vibration) of the system if the excitation force on $m_1$ was $\mathrm{F}(t) = 200\ \cos(15t)$. (5p)

**Question 2 (5 p)**



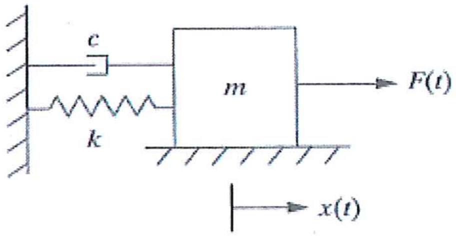

The mass shown in the figure, is attached to a moving base with a spring and a damper. The mass $m$ is 0.1 kg and the stiffness of the spring is $k = 4000\frac{\mathrm{N}}{\mathrm{m}}$.

a) When the base is oscillated with movement described by $y = 0.01\sin(200t)$, (in meters), the steady state oscillation amplitude of mass $m$ becomes 0.05 meters. Calculate the damping ratio of the system and damper's coefficient $c$. (2p)

b) Calculate the amplitude of the total force applied to the mass (in steady state vibration, after transient vibrations die out), when the base moves with $y = 0.01\sin(600t)$, if the damping ratio of the system remains the same as the one you calculated at part a. (2p)

c) describe the equation of motion of mass $m$ in response to the base motion with $y = 0.01\sin(600t)$, **in steady state vibration** (i.e. ignore the transient response). You need to specify the amplitude and phase of the motion. (1p)

## General Case:



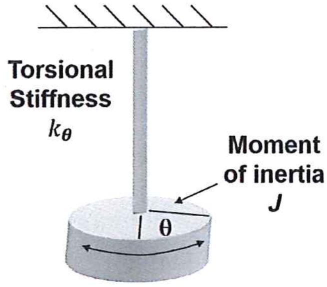

$m\ddot{x} + c\dot{x} + kx = F(t)$ Newton's 2nd Law (NSL)

with $\zeta = \frac{c}{2\sqrt{k.m}}, \omega_n = \sqrt{k/m}, f(t) = F(t)/m$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ Mass normalized form

initial conditions: $x(0) = x_0$ and $\dot{x}(0) = v_0$

### Dynamics of rotating systems




$$J[kg \cdot m^2]\,\ddot{\theta} + c_\theta\left[\frac{N \cdot m}{s}\right]\dot{\theta} + k_\theta[N \cdot m]\theta = T(t)[N \cdot m]$$

with $\zeta = \frac{c_\theta}{2\sqrt{k_\theta . J}}, \omega_n = \sqrt{\frac{k_\theta}{J}}, f(t) = T(t)/J$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)$$

# 1 Free Vibration (F(t) = 0)

## 1.1 Undamped Case (c = 0)

$$\ddot{x} + \omega_n^2 x = 0$$

**solution:**

| | | |
|---|---|---|
| $x(t) = a.\sin(\omega_n t) + b.\cos(\omega_n t)$ | $a = \frac{v_0}{\omega_n}$ | $b = x_0$ |
| | Or | |
| $x(t) = A.\sin(\omega_n t + \phi)$ | $A = \frac{\sqrt{x_0^2\omega_n^2 + v_0^2}}{\omega_n}$ | $\phi = \arctan\left(\frac{x_0\omega_n}{v_0}\right)$ |

## 1.2 Damped Case (With Viscous Damping, $c \neq 0$)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \zeta = \frac{c}{2\sqrt{k.m}} \neq 0$$

### 1.2.1 Underdamped case ($0 < \zeta < 1$)

**solution:**

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

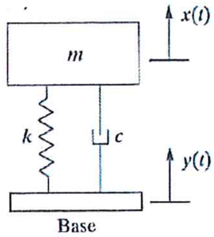
| | | |
|---|---|---|
| $x(t) = e^{-\zeta\omega_n t}[a.\sin(\omega_d t) + b.\cos(\omega_d t)]$ | $a = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$ | $b = x_0$ |
| | Or | |
| $x(t) = Ae^{-\zeta\omega_n t}.\sin(\omega_d t + \phi)$ | $A = \frac{\sqrt{\omega_d^2 x_0^2 + (v_0^2 + \zeta\omega_n x_0)^2}}{\omega_d}$ | $\phi = \arctan\left(\frac{\omega_d x_0}{v_0 + \zeta\omega_n x_0}\right)$ |

### 1.2.2 Critically damped case ( $\zeta = 1$)

| | | |
|---|---|---|
| $x(t) = (a_1 + a_2 t)e^{-\omega_n t}$ | $a_1 = x_0$ | $a_2 = v_0 + \omega_n x_0$ |
| Eq. 1.45 | Eq. 1.46 | Eq. 1.46 |

# 3 Base Excitation

## 3.1 Harmonic Excitation




$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0, \quad y = Y\sin(\omega t)$ from NSL

Standard form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega Y \times \cos(\omega t) + \omega_n^2 Y \times \sin(\omega t)$$

Particular solution:

$$x_p(t) = X\,.\sin(\omega.t - \psi)$$

$$X = Y.\sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

Or

$$X = Y.\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}, \qquad r = \frac{\omega}{\omega_n}$$

$$\psi = \arctan\left[\frac{mc\omega^3}{k(k - m\omega^2) + (c.\omega)^2}\right] = \arctan\left[\frac{2\zeta r^3}{(1 - r^2) + (2\zeta r)^2}\right]$$

$\left|\frac{X}{Y}\right| =.\sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **displacement transmissibility** and reaches to its maximum (resonance) very close to $r = 1$. This ratio reduces and reaches 0 as the r increases (i.e. when base vibration frequency increases to values much higher than the natural frequency, the mass remains almost still. In other words, you cannot oscillate an object at frequencies much higher than the natural frequency that is created between that object and its base). (Fig. 2.14)

### 3.1.1 Transmitted Force

$$F(t) = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x} = -m\omega^2 X.\sin(\omega.t - \psi)$$

$$\left|\frac{F}{Y}\right| = m\omega^2 \left|\frac{X}{Y}\right| = (k.r^2) \times \sqrt{\frac{1 + (2.\zeta.r)^2}{(1 - r^2)^2 + (2.\zeta.r)^2}}.$$

$\left|\frac{F}{kY}\right| = r^2 \times \sqrt{\frac{1+(2.\zeta.r)^2}{(1-r^2)^2+(2.\zeta.r)^2}}$ is called **force transmissibility ratio**. This function also has a local peak very close to $r = 1$ With non-zero damping ratio it keeps increasing as the $r$ ratio increases. (Fig. 2.15 in the book).

8. Multiply , $\boldsymbol{r}(\boldsymbol{t})$ by $\boldsymbol{S}$ to get the solution $x(t) = \boldsymbol{S}\,\boldsymbol{r}(\boldsymbol{t})$
   Note that $\boldsymbol{S}$ is the matrix of mode shapes and $\boldsymbol{P}$ is the matrix of eigenvectors.

### 7.1.2 General mass matrix, by Choleskhy decomposition

$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{0}$ *(eq. 4.54)*

1. Calculate lower triangular matrix $\boldsymbol{L}$ such that $\boldsymbol{M} = \boldsymbol{L}\boldsymbol{L}^T$ .
   a. If you can easily calculate $\boldsymbol{M}^{\frac{1}{2}}$, (e.g. when you have a diagonal $\boldsymbol{M}$ matrix), then you can replace $\boldsymbol{L}$ by $\boldsymbol{M}^{\frac{1}{2}}$ in the remining of equations. [this will result in 6.1].
2. Calculate $\boldsymbol{L}^{-1}$
3. Calculate the mass normalized stiffness matrix $\widetilde{\boldsymbol{K}} = \boldsymbol{L}^{-1}\boldsymbol{K}(\boldsymbol{L}^{-1})^T$
   Calculate the symmetric eigenvalue problem for $\widetilde{\boldsymbol{K}}$ to get $\omega_i^2$ and Orthonormal eigenvectors $\boldsymbol{v}_i$. Build $\boldsymbol{P}$ with these orthonormal eigenvectors:

$$\boldsymbol{P} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots ]$$

**[since** $\widetilde{\boldsymbol{K}}$ is a symmetric matrix its eigenvectors will be orthogonal to each other, i.e. $\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0$, But $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots \boldsymbol{v}_n$ should be normalized, i.e. their norm $\|\boldsymbol{v}_i\|$, square root of sum of square of elements, should be 1. You can ensure this by updating the eigenvectors as $\boldsymbol{v}_{i_{updated}} = \boldsymbol{v}_i/\sqrt{\|\boldsymbol{v}_i\|}$

Since the columns of $\boldsymbol{P}$ are orthonormal eigenvectors of $\widetilde{\boldsymbol{K}}$, then $\boldsymbol{P}^T\boldsymbol{P} = \mathrm{I}_{n\times n}$ ($n \times n$ unity matrix) and $\boldsymbol{P}^T\widetilde{\boldsymbol{K}}\,\boldsymbol{P} = \boldsymbol{\Lambda}$ .

and $\boldsymbol{\Lambda}$ is a diagonal matrix with square of natural frequencies for each modes shape as its main diagonal:

$$\Lambda = \mathrm{diag}(\omega_i^2) = \begin{bmatrix} \omega_1^2 & 0 & & \cdots & & \\ 0 & \omega_2^2 & & & & \\ & & \ddots & & & \\ & & & \omega_i^2 & & \\ & & & & \ddots & \\ & & & & & \omega_n^2 \end{bmatrix}$$

4. Calculate $\boldsymbol{S} = (\boldsymbol{L}^{-1})^T\,\boldsymbol{P}$ and $\boldsymbol{S}^{-1} = \boldsymbol{P}^T\boldsymbol{L}^T$
5. Calculate the modal initial conditions, $\boldsymbol{r}(\boldsymbol{0}) = \boldsymbol{S}^{-1}\boldsymbol{x}_0, \dot{\boldsymbol{r}}(\boldsymbol{0}) = \boldsymbol{S}^{-1}\dot{\boldsymbol{x}}_0$
6. Substitute , $\boldsymbol{r}(\boldsymbol{0})$ and $\dot{\boldsymbol{r}}(\boldsymbol{0})$ into equations (4.66) and (4.67) to get the solution in modal coordinate $\boldsymbol{r}(\boldsymbol{t})$:

$$r_i\,(t) = \frac{\sqrt{\omega_i^2 r_{i,0}^2 + \dot{r}_{i,0}{}^2}}{\omega_i}\sin\left(\omega_i t + \arctan\frac{\omega_i r_{i,0}}{\dot{r}_{i,0}}\right), i = 1, 2, \dots$$

7. Multiply , $\boldsymbol{r}(\boldsymbol{t})$ by $\boldsymbol{S}$ to get the solution $x(t) = \boldsymbol{S}\,\boldsymbol{r}(\boldsymbol{t})$
   Note that $\boldsymbol{S}$ is the matrix of mode shapes and $\boldsymbol{P}$ is the matrix of eigenvectors of $\widetilde{\boldsymbol{K}}$.

### 7.1.3 With diagonal mass matrix

You can use the same equations as 6.1.2, but use $\mathbf{L} = \mathbf{M}^{\frac{1}{2}}$ and $\mathbf{L}^{-1} = \mathbf{M}^{-1/2}$

## 7.2 Modal Analysis of the Forced Response, with general mass matrix and damping

$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{C}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{B}\boldsymbol{F}(\boldsymbol{t})$ *(eq. 4.126)*

$\boldsymbol{B}\boldsymbol{F}(\boldsymbol{t})$ is used to shape application of various force functions on degrees of freedom.

1. Calculate lower triangular matrix $\boldsymbol{L}$ such that $\boldsymbol{M} = \boldsymbol{L}\boldsymbol{L}^T$ .

Provided that the damping matrix has specific conditions, e.g. it is a proportional to mass and stiffness matrices as:

$$\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$$

## 7.3 Physical, Mass Normalized and Modal Spaces

| Line | Name | Mass Matrix | Damping Matrix | Stiffness Matrix | Needed Transformation | State Vector | Needed Transformation ↓ | Needed Transformation ↑ | Force Vector |
|---|---|---|---|---|---|---|---|---|---|
| 1 | Physical Space | $\mathbf{M}$ | $\mathbf{C}$ | $\mathbf{K}$ | | $\mathbf{X}(t)$ | | $\mathbf{X}(t) = (\mathbf{L}^{-1})^T\mathbf{q}(t)$ $= \mathbf{S}\,\mathbf{r}(t)$ | $\mathbf{BF}(t)$ |
| 2 | Mass Normalized | $\mathbf{I}$ | $\tilde{\mathbf{C}}$ | $\tilde{\mathbf{K}}$ | $\tilde{\mathbf{K}} = (\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T$ | $\mathbf{q}(t)$ | $\mathbf{q}(t) = \mathbf{L}^T\mathbf{X}(t)$ | $\mathbf{q}(t) = \mathbf{Pr}(t)$ | $\mathbf{L}^{-1}\mathbf{BF}(t)$ |
| 3 | Modal Space | $\mathbf{I}$ | $[diag(2\zeta_i\,\omega_i)]$ (*) | $\Lambda = [diag(\omega_i^2)]$ | $\Lambda = \mathbf{P}^T\tilde{\mathbf{K}}\mathbf{P}$ $= \mathbf{P}^T(\mathbf{L}^{-1})\,\mathbf{K}(\mathbf{L}^{-1})^T\mathbf{P}$ $= \mathbf{S}^T\mathbf{K}\,\mathbf{S}$ | $\mathbf{r}(t)$ | $\mathbf{r}(t) = \mathbf{P}^T\mathbf{q}(t)$ $= \mathbf{S}^{-1}\mathbf{X}(t)$ | | $\mathbf{f}(t)$ $= \mathbf{P}^T(\mathbf{L}^{-1})\mathbf{BF}(t)$ $= \mathbf{S}^T\mathbf{BF}(t)$ |

* Only possible if $\mathbf{S}^T\mathbf{C}\,\mathbf{S}$ becomes a diagonal matrix $[diag(2\zeta_i\,\omega_i)]$, e.g. when $\mathbf{C} = \alpha\mathbf{M} + \beta\,\mathbf{K}$, then $\mathbf{S}^T\mathbf{C}\,\mathbf{S} = \alpha\mathbf{I} + \beta\Lambda = [diag(2\zeta_i\,\omega_i)]$

**Transformation Matrices:**

| | Description | Definition | Calculation in MATLAB | With Diagonal $\mathbf{M}$ |
|---|---|---|---|---|
| $\mathbf{L}$ | Normalization of Mass Matrix Lower triangular Cholesky's matrix for $\mathbf{M}$ | $\mathbf{M} = \mathbf{L}\mathbf{L}^T$ | `L = chol(M, 'lower');` | $\mathbf{L} = \mathbf{M}^{1/2}$ |
| $\mathbf{P}$ | Makes $\tilde{\mathbf{K}}$ diagonal | $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots]$ Matrix of Orthonormal Eigenvectors of $\tilde{\mathbf{K}}$ | `K_tilde = (L^(-1))*K*(L^(-1))';` `[P, Lambda] = eig(K_tilde)` | |
| $S$ | Matrix of Mode Shapes, Moves from Modal Space to Physical Space | $\mathbf{S} = (\mathbf{L}^{-1})^T\mathbf{P}$ Also $\mathbf{S}^{-1} = \mathbf{P}^T\mathbf{L}^T$ and $\mathbf{S}^T = \mathbf{P}^T(\mathbf{L}^{-1})$ (in general, $\mathbf{S}^T \neq \mathbf{S}^{-1}$) | `S = (L^(-1))' * P` `% Or` `S = (L')\P` | |