Institutionen för ingenjörsvetenskap

TENTAMEN

Kurs   Diskret matematik (Discrete Mathematics)

Delkurs   Salstentamen (Written-Exam)

Kurskod   MA126 G1N   Högskolepoäng för tentamen   5,0

Datum   2025-03-25   Skrivtid 14.15-19.30

Ansvarig lärare   Yohannes Aklilu

Berörda lärare

Hjälpmedel/bilagor   Studentens miniräknare, högskolans miniräknare, Befogat formelblad

Övrigt

Anvisningar

☐ Ta nytt blad för varje lärare
☒ Ta nytt blad för varje ny fråga
☒ Skriv endast på en sida av papperet.
☒ Skriv namn och personnummer på samtliga inlämnade blad.
☒ Numrera lösbladen löpande.
☒ Använd inte röd penna.
☒ Markera med kryss på omslaget vilka uppgifter som är lösta.

Poänggränser   U-betyg(Fail grade): Not fulfilling the passing criterion.
G-betyg(Pass grade): At least 18 points.
VG-betyg (Pass with distinction grade): At least 28 points.

Skrivningsresultat bör offentliggöras inom 18 arbetsdagar

Lycka till!

Antal sidor totalt

Examination Instructions

The exam is assessed with a grade of *Pass with distinction* (VG), *Pass* (G) or *Fail* (U) based on how well your solutions demonstrate that the grading criteria for the course objectives have been met. Each task can give up to 6 points, a total of 36. For G-grade, a total of at least 18 points is required, for VG-grade at least 28.

To pass the examination with G-grade: Your answer to the tasks must be concise, but sufficiently detailed and formulated so that the line of thought can be easily followed. Some degree of calculation error can be acceptable, as long as the layout and motivation is correct. An answer, for example, with out any motivation, however, will not be accepted. Numerical values can be entered as expressions, suitably simplified, where root expressions, logarithms and exponential expressions can be included in addition to *pure numbers*, if needed.

In order to pass the exam with distinction (VG-grade): Your answer is required to be well-grounded and followed well-constructed mathematical reasoning that leads to a correct answer or conclusion. The answers must be well formulated and analysed, and draw relevant conclusions about the nature of the solutions in a logical and consistent manner.

The following course goals will be assessed on this examination:

- explain the central concepts and methods in discrete mathematics treated in the course,

- show good familiarity with integers and modular arithmetic, and in particular show some familiarity with the Euclidian algorithm, Fermat's little theorem, Euler's theorem and solve problems where they may be used,

- describe different graph theoretical relations, algorithms and their applications, e.g. graph searching, Kruskal's and Dijkstra's algorithms,

- analyze discrete mathematical structures and determine their characteristics using mathematical reasoning; typical examples of such structures are relations, graphs and trees,

- identify problems which can be solved by methods from discrete mathematics, and choose suitable methods and apply them in a structured way.

Good Luck!
YA

Examination tasks

1. Let the universal $\mathbb{U}$ set be the set of all positive integers ≤ 42. Given the sets

$$A = \{x \in \mathbb{U} \mid \gcd(x, 42) = 1\} \quad and \quad B = \{y \in \mathbb{U} \mid y \cong 3 \pmod 4\}.$$

(2p/task)

   (a) Are the following statements True or False? Justify your answer.

      i. The set $(A \setminus B) \cup (B \setminus A)$ contains 14 distinct elements.

      ii. All elements of A are primes, but not all primes < 42 belong to A.

   (b) Choose any element $x \in A$ such that $x > 1$ and find its inverse modulo 42.

   (c) Define an surjective (onto) function $f : A \to B$, or justify why it is not possible to define such a function.
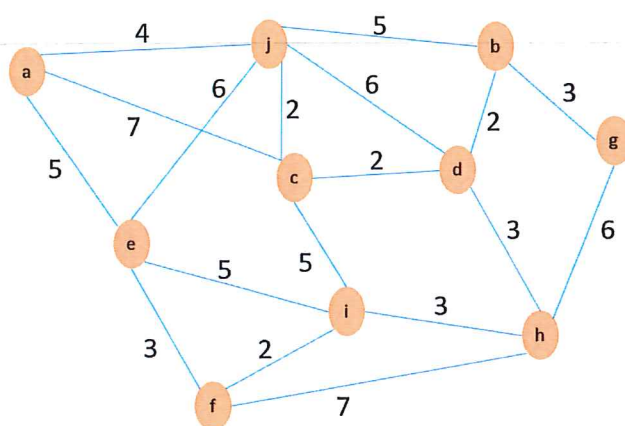
2.   (2p/task)

   (a) Let p, q and r be logical statements. Assume that the expressions $q \to p$ and $r \vee p$ are both False. Find the truth value of the following logical expression:

$$((\neg r \vee p) \to q)) \leftrightarrow (\neg p \to (r \wedge \neg q)).$$

   (b) Given the following weighted graph.

Figur 1: Task 2b

      i. Find a minimal spanning tree for the graph.

      ii. Determine the shortest path from node a to h.

3. You are given the following algebraic expression   (2p/task)

$$\left(\frac{3+\frac{1}{2}}{2-\frac{1}{3}}\right) + \left(\left(\frac{2}{5}+\frac{3}{7}\right)\cdot\frac{1}{4}\right).$$

(a) Represent the algebraic expressions with a binary tree.

(b) Use Breadth-First Search to traverse the binary tree.

(c) Determine the post-order traverse for the binary tree.

4. (a) Find the last digit to the number 3^{452}.   (2p)

(b) Convert the integer $(4523)_{eight}$ given in octal form to decimal form.   (2p)

(c) Solve the congruence equation $21x \cong 33 \pmod{48}$.   (2p)

5. (a) A graph G has the nodes $V = \{a, b, c, d, e, f, g, h, i\}$ and adjacency matrix, where the natural ordering of alphabets are used to construct the matrix,

$$G_M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(4p)

   i. Justify whether the graph is simple and undirected.

   ii. Find the number of paths of length 2 between nodes b and g.

(b) Construct a connected undirected graph with 10 nodes where the degree of the nodes are 3, 2, 4, 6, 7, 5, 6, 4, 8, and 2. If it is not possible, justify why. (2p)

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6. (a) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and R be a relation defined on the set $A \times A$ by

(a, b) is related to (c, d) iff $a + d = b + c$.

(2p/task)

    i. Prove that R is an equivalence relation.

    ii. Find the equivalence classes of $(2, 5)$.

(b) The congruence equation   (2p)

$$\begin{cases} x \equiv 3 \bmod 5 \\ x \equiv 7 \bmod 8 \\ x \equiv 4 \bmod 9 \end{cases}$$

has a solution modulus m, where m is an integer that can be calculated. Justify why it is possible to calculate, and determine the value of m in the current case.

Logic

p	$\neg p$
1	0
0	1

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Some set theoretic and logical relations or properties

	Set theoritical rules	**Logical Equivalences**
Dubble Complement	$(A^c)^c = A^{cc} = A$	$\neg\neg p \Leftrightarrow p$
Complement/negation	$A \cap A^c = \emptyset$	$p \wedge \neg p \Leftrightarrow 0$
	$A \cup A^c = U$	$p \vee \neg p \Leftrightarrow 1$
Idempotent	$A \cap A = A$	$p \wedge p \Leftrightarrow p$
	$A \cup A = A$	$p \vee p \Leftrightarrow p$
Commutative laws	$A \cap B = B \cap A$	$p \wedge q \Leftrightarrow q \wedge p$
	$A \cup B = B \cup A$	$p \vee q \Leftrightarrow q \vee p$
Associative laws	$A \cap (B \cap C) = (A \cap B) \cap C$	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
	$A \cup (B \cup C) = (A \cup B) \cup C$	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
DeMorgan rules	$(A \cup B)^c = A^c \cap B^c$	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
	$(A \cap B)^c = A^c \cup B^c$	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
	$A \backslash B = A \cap B^c$	$p \rightarrow q \Leftrightarrow \neg p \vee q$

Definitions and some operations on logic and set theory

Power set	$2^A = \{B \mid B \subseteq A\}$	
Cartesian product	$A \times B = \{(a,b) \mid a \in A \ \text{and} \ b \in B\}$	
Quantifiers	$\exists x$	There exists x
	$\forall x$	For all x
	$\neg\big(\forall x P(x)\big) \Leftrightarrow \exists x(\neg P(x))$	
	$\neg\big(\exists x P(x)\big) \Leftrightarrow \forall x(\neg P(x))$	

Matrices

Given matrices (or Boolean matrices) $A = (a_{ij})$ and $B = (b_{ij})$. We have the following operations as long as compatibility criterions are satisfied:

Matrix addition/subtraction	$A \pm B = (a_{ij} + b_{ij})$
Matrix multiplication	$A \cdot B = (c_{ij})$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$
Boolean conjunction	$A \wedge B = (a_{ij} \wedge b_{ij})$
Boolean disjunction	$A \vee B = (a_{ij} \vee b_{ij})$
Boolean product	$A \odot B = (c_{ij})$, where $c_{ij} = \big(a_{i1} \wedge b_{1j}\big) \vee \big(a_{i2} \wedge b_{2j}\big) \vee \cdots \vee \big(a_{ik} \wedge b_{kj}\big)$

Some properties:

Commutative rule	$A + B = B + A$
Associative rules	$A + (B + C) = (A + B) + C$
	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$
Distributive rule	$A \cdot (B + C) = A \cdot B + A \cdot C$
Transpose and matrix operations	$(A \pm B)^t = A^t \pm B^t$
	$(A \cdot B)^t = B^t \cdot A^t$

Arithmetic

The following rules of arithmetic apply for all integers a, b and c.

- Commutative rules: $a + b = b + a$ $a \cdot b = b \cdot a$
- Associative rules: $(a + b) + c = a + (b + c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Distributive rule: $a \cdot (b + c) = a \cdot b + a \cdot c$
- $(a \pm b)^2 = a^2 \pm 2ab + b^2$ $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
- $a^2 - b^2 = (a - b)(a + b)$ $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

The second-degree equation $ax^2 + bx + c = 0$, where $a \neq 0$ has a solution $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

Modular arithmetic

If $a_1 \cong b_1 (mod\ n)$ and $a_2 \cong b_2 (mod\ n)$, then	If $a \cong b(mod\ n)$ and $m \in \mathbb{Z}$, then
• $(a_1 + a_2) \cong (b_1 + b_2)(mod\ n)$. • $(a_1 - a_2) \cong (b_1 - b_2)(mod\ n)$. • $(a_1 \cdot a_2) \cong (b_1 \cdot b_2)(mod\ n)$.	• $m \cdot a \cong m \cdot b(mod\ n)$. • $a^m \cong b^m (mod\ n)$ for all integers $m \geq 0$.

Euler totient function

If $p_1, p_2, \ldots, p_k$ are the prime factors of n, then $\phi(n) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$.

The Chinese remainder theorem:

If $m_i, \ldots, m_k$ are pairwise relatively prime integers, then the system of linear equations

$$\begin{cases} x \cong a_1 \ (mod\ m_1) \\ x \cong a_2 \ (mod\ m_2) \\ \vdots \\ x \cong a_k \ (mod\ m_k) \end{cases}$$

have a solution $x \cong \sum_{i=1}^{k} a_i y_i M_i \ (mod\ M)$, where

- $M = \prod_{i=1}^{k} m_i,$
- $M_i = \frac{M}{m_i}$ *and*
- $y_i \cong M_i^{-1}(mod\ m_i)$

Relation and functions

Let $R: A \to A$ be a relation on a set A. Then R is

- *reflexive* if $(x, x) \in R$ for all $x \in A$.
- *symmetric* if $(x, y) \in R \to (y, x) \in R$.
- *antisymmetric* if $[(x, y) \in R \land (y, x) \in R] \to x = y$.
- *transitive* if $[(x, y) \in R \land (y, z) \in R] \to (x, z) \in R$.

Let $f: A \to B$ be a function from a set A to a set B. Then f is

- *injective* (one-to-one) if $f(a_1) = f(a_2) \to a_1 = a_2$.
- *surjective* (onto) if $(\forall b \in B)(\exists a \in A)(f(a) = b)$.
- *bijective* (one-to-one correspondence) if f is both injective and surjective.